BMT to AIME - 2025 Edition

Instructions:

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Benji writes down the decimal representation of a rational number greater than 10. He observes that if he replaces the tens digit with 2, the number decreases by 70%. If he instead replaces the tens digit with 8, the number increases by 14%. By what percent would the number decrease if he replaced the tens digit with 5?

2. The boxes in the expression below are filled with the numbers 1, 2, 3, 4, 5, 6, 7 and 8 so that each number is used exactly once and the value of the expression is positive. What is the least possible value of the expression?

$$\square \times \square \times \square \times \square - \square \times \square \times \square \times \square$$

3. In how many ways can ten identical white blocks be placed into a red box, a yellow box, a green box, and a blue box such that no box is empty, and at least two boxes contain the same number of blocks?

4. A *substring* of a positive integer n is a positive integer whose digits appear consecutively in the digits of n. For example, 10, $00 = 0$, and $02 = 2$ are two-digit substrings of 1002, but 12 and $01 = 1$ are not. Find the number of four-digit numbers such that each of its two-digit substrings are divisible by 3.

5. Quadrilateral $ABCD$ has side lengths $AB = BC = CD = 25$ and acute angles at vertices B and C. The distance from point C to side $\overline{AB}$ is 24, and the distance from point B to side $\overline{CD}$ is 20. Compute the area of quadrilateral $ABCD$.

6. Over all triples of positive integers (a, b, c) satisfying

$$\gcd(\operatorname{lcm}(a, b), \operatorname{lcm}(b, c), \operatorname{lcm}(c, a)) = 360,$$

find the least possible value of $a + b + c$.

7. A trading card store only sells small and large packs of cards, which cost \$11 and \$18, respectively. Each small pack has the same number of cards, each large pack has the same number of cards, and a large pack has more cards than a small pack. The greatest number of cards that can be bought with \$38 is 48, and the greatest number of cards that can be bought with \$56 is 75. What is the greatest number of cards that can be bought with \$202? In each scenario, not all the money needs to be spent.

8. In triangle ABC with $AB < AC$ and circumcircle ω, point M is the midpoint of side $\overline{BC}$. Line $\overleftrightarrow{AM}$ intersects ω at $D \neq A$. Let X and Y be points on ω such that $\overline{BX}$ and $\overline{CY}$ bisect $\overline{AM}$. Given $AM = 40$, $MD = 10$, and $BX = 50$, the length $CY = a\sqrt{b}$, where a and b are positive integers such that b is not divisible by the square of any prime. Find $a + b$.

9. Fourteen people are sitting at a circular table with fourteen distinct seats. They all get up to eat lunch and then return to the table. When they sit down again, each person sits either in their original seat or a seat adjacent to their original seat, and no two people sit in the same seat. In how many ways can this occur?

10. The polyhedron $ABCDEFG$ with seven vertices has six faces:

    - Two isosceles trapezoids $ABCD$ and $ABEF$ with $AB = 4$, $CD = EF = 1$, and $AD = AF = BD = BE = 3$,
    - Two triangles CDG and EFG with $CG = DG = EG = FG = \sqrt{3}$,
    - Two kites $ADGF$ and $BCGE$ with $\angle ADG = \angle AFG = \angle BCG = \angle BEG = 90°$.

    Find the square of the volume of the polyhedron $ABCDEFG$.

11. Let $\omega = e^{2i\pi/5}$. Find the remainder when

$$\prod_{i=1}^{5} \prod_{j=i+1}^{5} (\omega^i - \omega^j)^2$$

    is divided by 1000.

12. Let $N = 2025 \cdot 202^5 \cdot 20^{25}$. For each positive integer x, define $\sigma(x)$ to be the sum of the positive divisors of x. Define $h(x)$ to be the sum of all positive divisors y of x such that $\sigma(y)$ is even. For example, $\sigma(9) = 13$ and $h(9) = 3$. Let M be the number of positive divisors k of N such that $h(k)$ is odd. Find the remainder when M is divided by 1000.

13. Aarush and Brian are watching a sequence of fair coin flips. Both players start with 0 points. Aarush gains a point when the coin lands on heads, and Brian gains a point when the coin lands on tails. If Brian reaches 2 points, both players reset their points to 0. What is the expected number coin flips it takes for Aarush to reach 9 points for the first time?

14. Complex numbers $a_1, a_2, \ldots, a_{10}$ satisfy the following property:

$$\sum_{i=1}^{10} a_i^9 = 10a_k^{10} + a_k^9 - 2^9.$$

    Given $a_1 a_2 \cdots a_{10} = -\frac{m}{n}$, where m and n are relatively prime positive integers, compute the sum of the digits of $m + n$.

15. Triangle ABC has side lengths $AB = 6$ and $AC = 7$. Point D lies on side $\overline{BC}$ such that $AD = 3$. The internal angle bisector of $\angle BAD$ intersects the circumcircle of triangle BAD at point $X \neq A$, and the internal angle bisector of $\angle CAD$ intersects the circumcircle of triangle CAD at point

$Y \neq A$. Given that $\overline{XY} \parallel \overline{BC}$, the length $XY = \dfrac{\sqrt{m}}{n}$, where m and n are positive integers such that m is not divisible by the square of any prime. Find $m + n$.