BMT to AIME - 2023 Edition

Instructions:

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Consider two geometric sequences $16, a_1, a_2, \ldots$ and $56, b_1, b_2, \ldots$ with the same common nonzero ratio. Given that $a_{2023} = b_{2020}$, compute $b_6 - a_6$.

2. Let rectangle $ABCD$ have side lengths $AB = 8, BC = 6$. Let $ABCD$ be inscribed in a circle with center O. Let M be the midpoint of side $\overline{AB}$, and let X be the intersection of ray $\overrightarrow{MO}$ with the circle. Find AX^2.

3. Given positive integers $a \geq 2$ and k, let $m_a(k)$ denote the remainder when k is divided by a. Compute the number of positive integers, n, less than 500 such that $m_2(m_5(m_{11}(n))) = 1$.

4. Compute the sum of all positive integers n for which there exists a real number x satisfying
$$\log_2 \left(x + \frac{n}{x}\right)^n = 20.$$

5. Kait rolls a fair 6-sided die until she rolls a 6. If she rolls a 6 on the Nth roll, she then rolls the die N more times. The probability that she rolls a 6 during these next N times is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

6. Triangle $\triangle ABC$ has side lengths $AB = 8, BC = 15$, and $CA = 17$. Circles ω_1 and ω_2 are externally tangent to each other and within $\triangle ABC$. The radius of circle ω_2 is four times the radius of circle ω_1. Circle ω_1 is tangent to $\overline{AB}$ and $\overline{BC}$, and circle ω_2 is tangent to $\overline{BC}$ and $\overline{CA}$. The radius of circle ω_2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

7. Given a positive integer k, let $s(k)$ denote the sum of the digits of k. Let $a_1, a_2, a_3, \ldots$ denote the strictly increasing sequence of all positive integers n such that $s(7n + 1) = 7s(n) + 1$. Compute the remainder when a_{2023} is divided by 999.

8. Let $\triangle ABC$ have centroid G. If $\overline{GA} \perp \overline{GB}$, $AB = 12$, and $\angle ACB = 30^\circ$, then $AC \cdot BC = a\sqrt{b}$ for positive integers a and b such that b is squarefree. Find $a + b$.

9. Let N be the number of positive integers x less than $210 \cdot 2023$ such that
$$\operatorname{lcm}(\gcd(x, 1734), \gcd(x + 17, x + 1732))$$
divides 2023. Then $N = p^a \cdot q^b \cdot r^c$, where p, q, r are distinct prime numbers and a, b, c are positive integers. Find $p \cdot a + q \cdot b + r \cdot c$.

10. Maria and Skyler have a square-shaped cookie with a side length of 30 inches. They split the cookie by choosing two points on distinct sides of the cookie uniformly at random and cutting across the line segment formed by connecting the two points. If Maria always gets the larger piece, find the expected amount of extra cookie in Maria's piece compared to Skyler's, in square inches.

11. A tetrahedron has three edges of length 2 and three edges of length 4, and one of its faces is an equilateral triangle. The surface area of the sphere that is tangent to every edge of this tetrahedron is $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

12. Let t be the least real number such that there exist constants a, b for which the roots of $x^3 - ax^2 + bx - \frac{ab}{t}$ are the side lengths of a right triangle. Then $t = a + \sqrt{b}$, where a and b are positive integers such that b is squarefree. Find $a+b$.

13. Shiori places seven books, numbered from 1 to 7, on a bookshelf in some order. Later, she discovers that she can place two dividers between the books, separating the books into left, middle, and right sections, such that:

    - The left section is numbered in increasing order from left to right, or has at most one book.
    - The middle section is numbered in decreasing order from left to right, or has at most one book.
    - The right section is numbered in increasing order from left to right, or has at most one book.

    In how many possible orderings could Shiori have placed the books? For example, $(2,3,5,4,1,6,7)$ and $(2,3,4,1,5,6,7)$ are possible orderings with the partitions $2,3,5|4,1|6,7$ and $2,3,4|1|5,6,7$, but $(5,6,2,4,1,3,7)$ is not.

14. Let triangle $\triangle ABC$ be acute, and let point M be the midpoint of $\overline{BC}$. Let E be on line segment $\overline{AB}$ such that $\overline{AE} \perp \overline{EC}$. Then, suppose T is a point on the other side of line BC as A such that $\angle BTM = \angle ABC$ and $\angle TCA = \angle BMT$. If $AT = 14, AM = 9$, and $\frac{AE}{AC} = \frac{2}{7}$, then find BC^2.

15. Let α denote the positive real root of the polynomial $x^2 - 3x - 2$. Compute the remainder when $\lfloor \alpha^{1000} \rfloor$ is divided by the prime number 997. ($\lfloor r \rfloor$ denotes the greatest integer less than r.)