BMT to AIME - 2021 Edition

Instructions:

- This is a 3-hour timed exam of 15 problems. Do not begin to work on the problems before starting a 3-hour timer, and stop working when 3 hours is up.
- All answers are integers from 000 to 999 (you may choose to include or exclude leading zeros for this test). Each correct answer is worth 1 point; an incorrect or blank answer is worth 0 points.
- No calculators or computational aids are allowed.
- Answers and problem sources will be provided; find the solutions by checking out the original problem from the BMT archives. Every problem is sourced from BMT (mostly the original with modified answer extraction, sometimes a slight variant).

1. Moor and Samantha are drinking tea at a constant rate. If Moor starts drinking tea at 8:00am, he will finish drinking 7 cups of tea by 12:00pm. If Samantha joins Moor at 10:00am, they will finish drinking the 7 cups of tea by 11:15am. The number of hours it would take for Samantha to drink 1 cup of tea is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. How many three-digit numbers $\underline{abc}$ have the property that when it is added to $\underline{cba}$, the number obtained by reversing its digits, the result is a palindrome? (Note that $\underline{cba}$ is not necessarily a three-digit number since before reversing, c may be equal to 0.)

3. An equilateral polygon has unit side length and alternating interior angle measures of 15° and 300°. The area of this polygon is $a+\sqrt{b}-\sqrt{c}$, where a, b, and c are positive integers. Find $a+b+c$.

4. Shivani has a single square with vertices labeled $ABCD$. She is able to perform the following transformations:

    - She does nothing to the square.
    - She rotates the square by 90, 180, or 270 degrees.
    - She reflects the square over one of its four lines of symmetry.

    For the first three timesteps, Shivani only performs reflections or does nothing. Then for the next three timesteps, she only performs rotations or does nothing. She ends up back in the square's original configuration. Compute the number of distinct ways she could have achieved this.

5. Compute the sum of all positive integers n such that n^n has exactly 325 positive integer divisors.

6. Let circles ω_1 and ω_2 intersect at P and Q. Let the line externally tangent to both circles that is closer to Q touch ω_1 at A and ω_2 at B. Let point T lie on segment $\overline{PQ}$ such that $\angle ATB = 90^\circ$. If $AT = 6, BT = 8$, and $PT = 4$, then $PQ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

7. Let z_1, z_2, and z_3 be the complex roots of the equation $(2z-3\overline{z})^3 = 54i+54$. When z_1, z_2, and z_3 are plotted in the complex plane, they are the vertices of a triangle with area $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is squarefree. Find $a+b+c$.

8. Dexter and Raquel are playing a game with N stones. Dexter goes first and takes one stone from the pile. After that, the players alternate turns and can take anywhere from 1 to $x+1$ stones from the pile, where x is the number of stones the other player took on the turn immediately prior. The winner is the one to take the last stone from the pile. Assuming Dexter and Raquel play optimally, compute the number of positive integers $N \le 2021$ where Dexter wins this game.

9. Let f be a real function such that for all $x \neq 0, 1$,

$$f(x) + f\left(-\frac{1}{x-1}\right) = \frac{9}{4x^2} + f\left(1 - \frac{1}{x}\right).$$

Then $f\left(\frac{1}{2}\right) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. Consider the randomly generated base 10 real number $r = 0.\underline{p_0}\,\underline{p_1}\,\underline{p_2}\ldots$, where each p_i is a digit from 0 to 9, inclusive, generated as follows:

    - $\underline{p_0}$ is generated uniformly at random from 0 to 9, inclusive.
    - For all $i \ge 0$, $\underline{p_{i+1}}$ is generated uniformly at random from $\underline{p_i}$ to 9, inclusive.

    The expected value of r is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

11. The line ℓ passes through vertex B and the interior of regular hexagon $ABCDEF$. If the distances from ℓ to the vertices A and C are 7 and 4, respectively, then the area of hexagon $ABCDEF$ is $a\sqrt{b}$, where a and b are positive integers such that b is squarefree. Find $a+b$.

12. Given an integer c, the sequence $a_0, a_1, a_2, \ldots$ is generated using the recurrence relation $a_0 = c$ and $a_i = a_{i-1}^i + 2021a_{i-1}$ for all $i \ge 1$. Let $f(c)$ be the smallest positive integer n such that $a_n - 1$ is a multiple of 47. Compute the remainder when

$$\sum_{k=1}^{46} f(k)$$

is divided by 1000.

13. Let $f(x) = x^3 - rx^2 + sx^2 - \frac{4\sqrt{2}}{27}$. The roots of f are the side lengths of a right triangle. The smallest possible area of this triangle is $\sqrt[3]{\frac{a}{b}}$, where a and b are relatively prime positive integers. Find $a+b$.

14. In $\triangle ABC$, let D and E be points on the angle bisector of $\angle BAC$ such that $\angle ABD = \angle ACE = 90^\circ$. Furthermore, let F be the intersection of lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{BC}$, and let O be the circumcenter of $\triangle AFC$. If $\dfrac{AB}{AC} = \dfrac{3}{4}$, $AE = 40$, and $\overline{BD}$ bisects $\overline{EF}$, then the perpendicular distance from A to $\overleftrightarrow{OF}$ is $\dfrac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is squarefree. Find $a + b + c$.

15. Let N be the number of ways to draw 22 straight edges between 10 labeled points, of which no three are collinear, such that no triangle with vertices among these 10 points is created, and there is at most one edge between any two labeled points. Find the remainder when N is divided by 1000.