Counting

We all learn to count as kids, but the study of counting in math has a lot more to it than learning the numbers 1-10. The study of counting objects and ways to obtain desired outcomes is called **combinatorics** (some people like to shorten this to *combo*). Combinatorics is also highly related to the study of **probability**, which is about finding the chance that something happens. (We'll talk about that in another worksheet!)

For now, one of the most crucial ideas in combinatorics is the **Counting Principle**, which is best explained by an example.

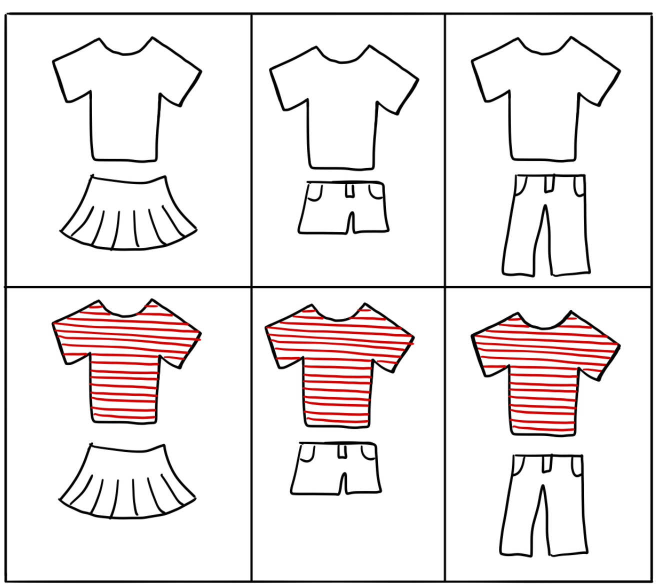
Let's say we have a new BmMT Activity - it's BmMT Fashion Week, and you want to be the contestant with the BEST outfit.

- Will you wear a striped or solid shirt?
- You have 3 options for your bottoms: a skirt, shorts, or jeans.
- Will you wear boots or sneakers?
- Will you wear pink socks, blue socks, red socks, or gray socks?

A natural question to ask is *how many possible outfits are there, given all these choices?*

With all the options, counting all of these might seem challenging (or time-consuming) if you don't know how to do it. A good problem-solving strategy when you don't know how to start a problem is to make it simple enough so that you can solve it, and then slowly add back the hard parts until you can solve the original problem.

For this problem, let's start by only thinking about the bottoms and the shirt. Whether you wear a striped shirt or solid shirt, either way you have 3 choices for the bottoms that you wear with it. This gives you a total of $2 \times 3 = 6$ different ways of choosing a shirt and bottoms. See the diagram below!



Notice how the in the table, the row corresponds to the choice of shirt and the column corresponds to the choice of bottoms. Do you see how this shows why multiplying 2×3 is the right thing to do?

Discuss: If we had a third shirt option, how would the table change, and how would the product change?

Soln: We'd need a new row for the new shirt, each column still represents the bottoms we choose. The table of possibilities would have three rows and three columns, which corresponds to 3×3.

So, we just multiplied the number of shirts by the number of bottoms to get the number 6. The Counting Principle says that if you make a sequence of choices with a fixed number of options for each choice, we can multiply the number of possibilities for each individual choice together to get the total number of possibilities.

This means for our outfit, there are

$$3 \times 2 \times 2 \times 4 = \boxed{48}$$

possible outfits to wear!

Question 0

You go to a bagel shop with four different types of bagels (plain, whole wheat, rye, and pumpernickel) and three types of topping (butter, cream cheese, and smoked salmon). If you can choose one bagel with one topping, how many different orders can you get?

Solution
Following the Counting Principle, there's 4 options for your first choice and 3 options for your second, so the answer is $4 \times 3 = \boxed{12}$.

What if you order for yourself and for your friend?

Solution
There are two ways to think about this problem. One way is to pick a bagel (4 options), pick a topping (3 options), pick another bagel (4 options), and pick another topping (3 options). This gives you a total of $4 \times 3 \times 4 \times 3 = \boxed{144}$.

However, if you already solved the first part, you know there are 12 ways to order a bagel. Then, if you order two times, you get $12 \times 12 = \boxed{144}$ options for you and your friend.

Question 1

Let's say you were stopping by the Berkeley mini Milk Treats ice cream shop, and decide to get their four scoop BmMT special. It's a mix and match deal, so you get a lot of choices:

- First, do you want it in a waffle cone or sugar cone?
- For the first scoop, do you want banana, blueberry, brownie, or butterscotch?
- For the second scoop, do you want mango, mint, mocha, or maple?
- For the third scoop, do you want macademia or marshmallow?
- For the fourth scoop, do you want toffee, taro, or tiramisu?
- Finally, do you want sprinkles or not?

How many different BmMT special orders are there?

Solution
Again, the choices don't depend on each other, so we can multiply the number of possibilities for each individual choice together and get the total number of possibilities. In this case, there are

$$2 \times 4 \times 4 \times 2 \times 3 \times 2 = \boxed{384}$$

possible BmMT ice cream specials.

(Unfortunately, we don't have plans at this time to actually serve 384 different ice cream specials, but maybe one day!)

Question 2

A phone number can be constructed as XXX-XXX-XXXX where the X's are any digit from 0 to 9. How many phone numbers are there total?

Solution

For each digit, we have 10 options. We have 10 digits to choose, so we have 10 multiplied by itself 10 times, so there are 10^{10} phone numbers.

What if you are told that the first number can't be a zero?

Solution

In this case, you have 9 options for the first digit, and 10 for the other 9 digits. So we'd get 9×10^9.

Question 3

If you have 5 different homework problems to do, how many different orders are there that you can do them in?

Solution

This one is trickier, because it doesn't seem like the choices are independent: if we do one problem, we can't go back and do it again later. However, the important thing is to make sure that no matter what we choose, we will have the same *number of options* to choose from in all of our separate choices. In this case, we can pick our first problem to do out of 5 possible problems. Then, no matter what we choose, we have 4 problems remaining to be the second problem we do, and then 3 problems to be the third problem, and so on. So, our final answer becomes

$$5 \times 4 \times 3 \times 2 \times 1 = 5! = \boxed{120}$$

When we write $n!$ (this is called n *factorial*), we mean the product of all the whole numbers less than or equal to n. In general, whenever you have to put n items in order without any other restrictions, there are $n!$ ways of doing it.