Sequences

Here's a question: if you had to cut a cake using only straight lines, what is the maximum number of pieces you could make with each cut?

Starting from the initial whole circle (cake!), any cut we make will cut the circle into two segments. To make the maximum number of pieces, the next cut has to cross all previous cut lines. Grab a piece of paper and try this for yourself - you should be able to create 2, 4, then 7, 11 segments and so on. A natural question then arises: is there a general formula for the maximum number of pieces you can have after n cuts? This is known as the *Lazy Caterer problem*, and to answer this question relies on analyzing the *sequence* $2, 4, 7, 11 \ldots$.

Sequences show up nearly everywhere in mathematics—whenever you have a "list" of things, chances are you can think of it as a sequence. Mathematicians like to analyze the properties and behaviors of sequences under different operations (e.g., adding consecutive terms, analyzing differences, combining different sequences, and many more) and in different contexts. For instance, from the sequence $1, 2, 3, \ldots$ we can consider the sum of consecutive terms, where each sum looks like

$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$\ldots$$

and continuing this process, we could ask

(Question 1) What is the sum of the terms $1 + 2 + 3 + \cdots + n$?

The first thing we can notice is that the list of sums $1, 3, 6, \ldots$ is also a sequence! In fact, this sequence is almost identical to the Lazy Caterer sequence and, as we just saw, this is in fact the sequence defined by the equation $1+2+3+\cdots+n$. This highlights a key property of sequences—the terms of a sequence are defined by an equation or "rule", although this may not be simple or even explicitly written out.

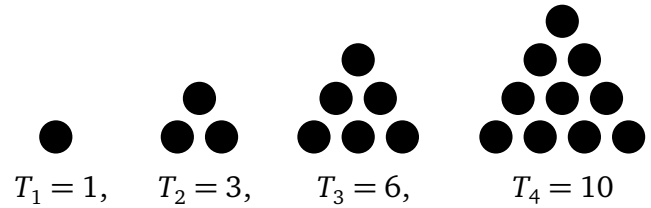


Figure 1: *The sums of the first n positive whole numbers, $T_n = 1 + 2 + \cdots + n$, are also called the 'triangular numbers' because they can be stacked in a triangle!*

Arithmetic Sequences

Now that we know what a sequence *is*, let's look at a few different kinds of sequences.

Imagine somebody gave you the sequence $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7, \ldots$. Could you find the formula for the n-th term, a_n, of this sequence? (pause and think!)

First, let's clarify the mysterious notation used above. Since sequences are really just lists defined by an equation, we can write a_n for the n-th item of this list, corresponding to the n-th term of the sequence. The little n is called a "subscript" or "index", which denotes which term in the sequence we are considering. The letter used for a subscript could be anything, but most commonly you might see authors use i, k, j, n, m; keep in mind that these all mean the same thing.

In this case, the sequence is just all the odd whole numbers and so the formula should reflect this—starting from $k = 1$, we can write $a_k = 2k - 1$. What about $2, 6, 10, 14, \ldots$? Hopefully it is clear that each term just increases by 4, and so a general formula should be $a_n = 4n - 2$. It turns out that both of these sequences are examples of the general concept of an **arithmetic sequence**.

> **(Definition)** An arithmetic sequence is a sequence defined by the equation $a_n = a_1 + (n-1)d$, where a_1 is the initial term of the sequence and d is called the 'common difference'.

(Question 2) Identify the initial term and common difference of the two examples given above. How can you rewrite the given formulae to match the general form?

(Question 3) If $a_1, a_2, \ldots, a_n$ is an arithmetic sequence of real numbers with common difference d, find a general formula for a_k in terms of a_j and d for some fixed term a_j in the sequence, where $1 \leq j \leq n$.

(Question 4) For an arithmetic sequence $a_1, \ldots, a_n$ of real numbers with common difference d, find a general formula for the sum

$$a_1 + a_2 + \cdots + a_n.$$

Use this to answer Question 1!

(BMT 2025, General Q13) Harsh writes an arithmetic sequence where the ratio of the second term to the first term is 4, and the sum of the third and fourth terms is 34. Compute the second term of Harsh's sequence.

Geometric Sequences

When looking at arithmetic sequences, we should understand by now that their key defining property is the linear relationship between consecutive terms. That is, for an arithmetic sequence $a_1, \ldots, a_n$ and any $1 < k \leq n$, $a_k - a_{k-1} = d$ where d is the common difference. There is another kind of sequence that is also "nice" to study: imagine you are given the sequence $1, 2, 4, 8, \ldots$. You can (and should) check that this does not fit into the formula of an arithmetic sequence. However, computing the ratio between consecutive terms, one can see that the ratio always remains constant. This suggests that the relationship between consecutive terms is multiplicative rather than additive. Such sequences are called **geometric sequences**.

Let's examine a few more examples—motivated students should try to create their own as well.

1. The sequence $1, 3, 9, 27, \ldots$
2. The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
3. The sequence $3, 6, 12, 24, \ldots$

By looking at the first two sequences, we can observe that if the first term a_1 is equal to 1, then the entire sequence is simple to understand: it *has* to be the consecutive powers of some fixed number! (What number is this for the first two examples given above?) This observation motivates the following definition:

(Definition) A geometric sequence $a_1, a_2, \ldots, a_n$ is a sequence defined by the equation $a_n = a_1 r^{n-1}$, where r is called the 'common ratio'.

(Question 5) Similar to arithmetic sequences, we can construct a new sequence from a geometric sequence by looking at sums of increasing length. For instance, the geometric sequence $1, 2, 4, 8, \ldots, 2^{n-1}$ has common ratio $r = 2$. By computing the first few sums $1, 1+2, 1+2+4, \ldots$, predict a formula for the general sum

$$1 + 2 + 4 + \cdots + 2^{n-1}.$$

(Question 6) Extending the ideas in Question 5, try to find the sum of the first n terms of a general geometric sequence $a_k = a_1 r^{k-1}$. (Figure 2 shows one such sum as a picture.)

(BmMT 2023, Team Round Q9) A geometric sequence with more than two terms has first term x, last term 2023, and common ratio y, where x and y are both positive integers greater than 1. An arithmetic sequence with a finite number of terms has first term x and common difference y. Also, of all arithmetic sequences with first term x, common difference y, and no terms exceeding 2023, this sequence is the longest. What is the last term of the arithmetic sequence?

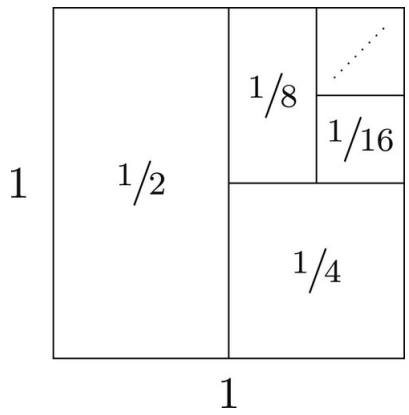


Figure 2: *A visual representation of summing the terms of the geometric sequence defined by* $a_1 = \frac{1}{2},\ r = \frac{1}{2}$