1. Timothe is reserving 100 classrooms for BmMT this year, and each room has exactly one purpose: testing, grading, or activities. Half of the rooms are for testing and five rooms are for grading. How many rooms are left for activities?

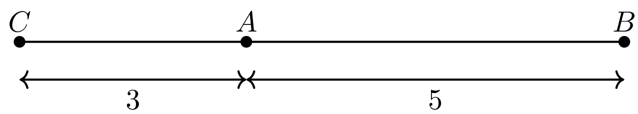
   Answer: 45

   Solution: Out of the 100 rooms, $\frac{1}{2} \cdot 100 = 50$ rooms are for testing, and 5 rooms are for grading. This leaves $100 - 50 - 5 = \boxed{45}$ rooms for activities.

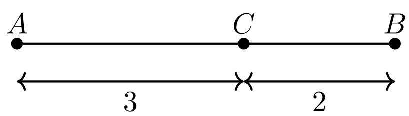
2. Points A, B, and C lie on a straight line, not necessarily in that order. The distance between A and B is 5, and the distance between A and C is 3. What is the sum of all distinct possible distances between B and C?

   Answer: 10

   Solution: First, draw points A and B such that they are 5 units apart, with A positioned to the left of B. Note that point C can lie on either side of A: either between A and B, or to the left of A. If C is to the left of A, then $BC = 8$.



   If C is between A and B (i.e., to the right of A), then $BC = 2$.



   The sum of all possible lengths BC is $2 + 8 = \boxed{10}$.

3. Benji writes down the first five odd positive integers and erases one of them. Kiran writes down the first five even positive integers and erases one of them. Benji notices that the sum of his four remaining numbers is equal to the sum of Kiran's four remaining numbers. If the number that Benji erased is B and the number that Kiran erased is K, find $K - B$.

   Answer: 5

   Solution:

   Before anyone erases anything, Benji has the numbers $\{1, 3, 5, 7, 9\}$ and Kiran has the numbers $\{2, 4, 6, 8, 10\}$.

   Suppose Benji erases the number B and Kiran erases the number K. The sum of the numbers Benji has left is $1+3+5+7+9-B$, and the sum of the numbers Kiran has left is $2+4+6+8+10-K$. Then

$$1 + 3 + 5 + 7 + 9 - B = 2 + 4 + 6 + 8 + 10 - K \implies 25 - B = 30 - K \implies K - B = 5.$$

   Thus, the **absolute** difference between the erased numbers is $\boxed{5}$.

4. A bag contains two red marbles, two blue marbles, and two green marbles. Richard removes marbles from the bag one at a time without replacement. What is the least number of marbles

Richard must remove to guarantee that two of the marbles that he has taken out are the same color?

Answer: 4

Solution: If Richard only removes 1 marble he can't possibly have 2 marbles of the same color.

Similarly, taking out 2 marbles doesn't guarantee Richard has taken 2 marbles of the same color because he could have picked 2 marbles of different colors, such as one red and one blue.

Even taking 3 marbles isn't enough, as Richard could take three marbles of different colors: one red, one green, and one blue.

With $\boxed{4}$ marbles picked, Richard is guaranteed to have taken a pair of marbles with the same color.

5. Four consecutive odd integers sum to -16. What is the product of these four integers?

Answer: 105

Solution 1: -16 is a negative number, so some or all of the odd integers are probably negative. We can play around with a few possibilities:

$$(-11) + (-9) + (-7) + (-5) = -32$$

is too small, and

$$(-5) + (-3) + (-1) + 1 = -8$$

is not small enough. We can experiment with groups of four consecutive odd integers in this range and see that

$$(-7) + (-5) + (-3) + (-1) = -16$$

satisfies our requirement. The product of these odd numbers is $(-7)(-5)(-3)(-1) = 15 \cdot 7 = \boxed{105}$.

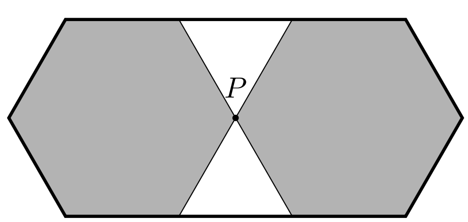
Solution 2: Let the four consecutive odd numbers be $2n-3, 2n-1, 2n+1, 2n+3$.

Then the sum $8n = -16$, which implies $n = -2$. This means the initial four numbers were $-7, -5, -3$, and -1.

The product is $(-7)(-5)(-3)(-1) = 15 \cdot 7 = \boxed{105}$.

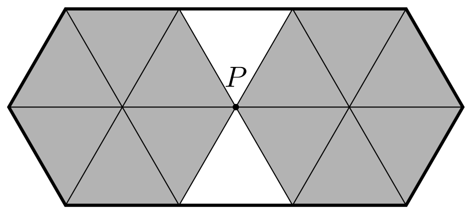
Solution 3: Let the numbers be $2n-3, 2n-1, 2n+1, 2n+3$. Then the sum $8n = -16, n = -2$. The product is $(4n^2-1)(4n^2-9)$, and plugging in $n = -2$ yields $(16-1)(16-9) = 15 \cdot 7 = \boxed{105}$.

6. Point P is a vertex shared by two congruent regular hexagons (both shaded) and two equilateral triangles (both unshaded) with the same side length, as shown below. Together, these shapes form a larger, non-regular hexagon (drawn with a thick border). What is the ratio of the **combined** area of both regular hexagons (the shaded area) to the area of the large hexagon? The answer may be expressed in any form.



Answer: $\frac{6}{7}$

Solution:



As shown above, each of the regular hexagons can be cut into 6 equilateral triangles. Since the larger hexagon is made of 2 regular hexagons and 2 equilateral triangles, it has a total area equal to that of $2 \cdot 6 + 2 = 14$ equilateral triangles. Thus, the ratio of the combined area of the two regular hexagons to the larger hexagon is $\frac{2 \cdot 6}{14} = \boxed{\frac{6}{7}}$.

7. How many positive factors of 120 are divisible by 12? Note that 1 and 120 are positive factors of 120.

Answer: **4**

Solution: Since any such factor must be a multiple of 12, we express 120 as:

$$120 = 12 \cdot 10.$$

This means any factor of 120 that is divisible by 12 can be written as:

$$12 \cdot (\text{some number}).$$

For this to still be a factor of 120, the "some number" must also divide 120 evenly. Rearranging, we see that this "some number" must be a factor of:

$$\frac{120}{12} = 10.$$

Thus, every factor of 120 that is divisible by 12 corresponds exactly to a factor of 10, multiplied by 12.

Now, finding the number of factors of 10:

$$10 = 2^1 \cdot 5^1.$$

Using the formula for the number of factors:

$$(1 + 1)(1 + 1) = 4.$$

So, 10 has 4 factors: $1, 2, 5$, and 10, which correspond to the factors of 120 divisible by 12:

$$12, 24, 60, 120.$$

Thus, the final answer is $\boxed{4}$.

8. Luke the frog lives on a pond with 5 lilypads, labeled 1 through 5. He starts at lilypad 1 and, at every step, hops to a lilypad with a larger number, chosen uniformly at random. Luke continues hopping until he reaches lilypad 5. What is the probability that it takes Luke exactly 2 steps to reach lilypad 5?

Answer: $\mathbf{\frac{11}{24}}$

Solution:

Let's consider all possible paths from lilypad 1 to lilypad 5 that take exactly two hops:

$$1 \to 2 \to 5$$

$$1 \to 3 \to 5$$

$$1 \to 4 \to 5$$

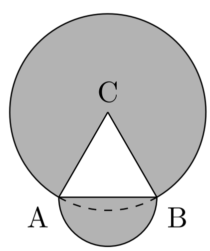
Now we'll determine the probability of Luke taking each of these paths.

The probability that Luke hops from lilypad 1 to lilypad 2 is $\frac{1}{4}$ since there are 4 lilypads (2, 3, 4, and 5) with number greater than 1, and Luke is equally likely to hop to 2, 3, 4, or 5. The probability that Luke then hops from lilypad 2 to lilypad 5 is $\frac{1}{3}$ since there are 3 lilypads with number greater than 2. Thus, the probability of the path $1 \to 2 \to 5$ is $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12} = \frac{2}{24}$.

By a similar analysis, we see the probability of the path $1 \to 3 \to 5$ is $\frac{1}{8} = \frac{3}{24}$ and the probability of $1 \to 4 \to 5$ is $\frac{1}{4} = \frac{6}{24}$.

Thus, the probability that Luke takes any of these paths is $\frac{2}{24} + \frac{3}{24} + \frac{6}{24} = \boxed{\frac{11}{24}}$.

9. Around equilateral triangle $\triangle ABC$ shown below, a circle centered at C with radius $\overline{CA}$ and a semicircle with diameter $\overline{AB}$ are drawn. If the total area of the shaded region (the whole shape excluding $\triangle ABC$) is 138π, what is the perimeter of $\triangle ABC$?



Answer: 36

Solution: Let the side length of the triangle be s. The radius of the semicircle is therefore $\frac{s}{2}$, so the area of the semicircle is $\frac{\pi s^2}{8}$. The radius of the large circular arc is s, and the fraction of the circle covered is $\frac{360-60}{360} = \frac{5}{6}$ because there is a 60^o arc left out (this angle measure is the internal angle of the equilateral triangle). Thus, the area of the large circular arc is $\frac{5\pi s^2}{6}$, and the sum of the two circular sectors' areas is $138\pi = \pi s^2 \left(\frac{1}{8} + \frac{5}{6}\right) \to s = 12$. Thus, the perimeter of the triangle is $s + s + s = \boxed{36}$.

10. Aaron has 1 penny, 1 nickel, 1 dime, and 1 quarter. These coins are worth 1, 5, 10, and 25 cents, respectively. Aaron makes a list of all the different numbers of cents he can make with some or all of his coins. If Aaron adds up all the numbers in his list, what is the result **in cents**?

Answer: 328

Solution 1:

We can approach the problem by considering different cases depending on whether the quarter is included, followed by subcases based on whether the dime is included, and so on. By listing all possible cases as shown below, we find that there are 16 distinct possible values in cents. The sum of these values is $\boxed{328}$.

Quarters	Dimes	Nickels	Pennies	Total (cents)
0	0	0	0	0
0	0	0	1	1
0	0	1	0	5
0	0	1	1	6
0	1	0	0	10
0	1	0	1	11
0	1	1	0	15
0	1	1	1	16
1	0	0	0	25
1	0	0	1	26
1	0	1	0	30
1	0	1	1	31
1	1	0	0	35
1	1	0	1	36
1	1	1	0	40
1	1	1	1	41

Solution 2:

It is important to note that there are $2^4 = 16$ subsets of the set $\{\text{quarter}, \text{dime}, \text{nickel}, \text{penny}\}$, since the set contains four items, and each item can either be included or excluded from the subset. The penny appears in exactly half of these subsets, as it is equally likely to be included or excluded. Therefore, the penny appears in $\frac{16}{2} = 8$ of the subsets. The same is true for the nickel, dime, and quarter. Additionally, no two subsets yield the same cent value. Thus, the sum of the cent values of all subsets is:

$$8 \cdot 1 + 8 \cdot 5 + 8 \cdot 10 + 8 \cdot 25 = 8 \cdot (1 + 5 + 10 + 25) = 8 \cdot 41 = \boxed{328}.$$

11. Find the sum of all numbers n such that the equation $x^2 - nx + 80 = 0$ has two positive integer solutions, and one solution is an integer multiple of the other.

Answer: 147

Solution: Let $f(x) = x^2 - nx + 80$. We can rewrite $f(x)$ as $f(x) = (x - r_1)(x - kr_1)$, where k is some positive integer. Expanding this out gives us

$$f(x) = x^2 - (k + 1)r_1 x + kr_1^2$$

This yields the following two equations:

$$(k + 1)r_1 = n$$

$$kr_1^2 = 80$$

Looking at $kr_1^2 = 80$, since both k and r_1 must be integers, and r_1^2 must divide 80, we have the solutions of $(k, r_1) = \{(80, 1), (20, 2), (5, 4)\}$. Therefore, the sum of all possible values of n is $81 \cdot 1 + 21 \cdot 2 + 6 \cdot 4 = \boxed{147}$.

12. A positive integer n is *two-cool* if the decimal expansion of $n/250$ has exactly two digits past the decimal point, excluding trailing zeros. For example, $10/250 = 0.04$ has exactly two digits past the decimal point, and $100/250 = 0.4$ has exactly one digit past the decimal point, so 10 is *two-cool* but 100 is not. How many positive integers less than 250 are *two-cool*?

    Answer: 40

    Solution 1: Note that any *two-cool* integer less than 250 (250 is not *two-cool*) divided by 250 has the form $0.xy$ where y is not zero. To count *two-cool* integers, consider working backwards, starting with a two-digit decimal $0.xy$ where $y \neq 0$. There are $10 \cdot 9 = 90$ such decimal numbers. Which of these correspond to *two-cool* integers? That is, when is $0.xy \cdot 250$ an integer?

    By experimenting with small numbers, we might find that $0.xy \cdot 250$ is a *two-cool* integer if y is even. To prove this, note that we must have $\frac{10x+y}{100} = \frac{n}{250} \implies \frac{5}{2}(10x + y) = 25x + \frac{y}{2} = n$ so y must be even. There are 4 even digits y could be: 2, 4, 6, or 8, and x can be any digit 0-9, so there are $10 \cdot 4 = \boxed{40}$ *two-cool* positive integers less than or equal to 250.

    Solution 2: Notice that $\frac{1}{250} = 0.004$ and the first *two-cool* integer appears at $n = 5$, since $\frac{5}{250} = \frac{1}{50} = 0.02$. Hence, all fractions that fulfill the *two-cool* condition can be represented as $\frac{n}{50}$. Of these n, we can see that any n that is a multiple of 5 will result in a decimal expansion of less than 2 digits.

    There are 50 possible values of n. Of these 50 values, 10 of them are multiples of 5 and thus cannot fulfill the *two-cool* condition. Therefore, the answer is $50 - 10 = \boxed{40}$

13. Points A, B, and C lie on a circle. Let segment $\overline{BC}$ extend through C to point D such that $\overline{AD}$ is tangent to the circle. If $AC = 4$, $BC = 9$, and $\angle ACD = 90^\circ$, what is CD?

    Answer: $\mathbf{\frac{16}{9}}$

    Solution: Since $\angle ACD = 90^\circ$, $\angle ACB = 90^\circ$. Then we conclude that $\overline{AB}$ is a diameter of the circle. Then for $\overline{AD}$ to be tangent to the circle, we must have $\angle BAD = 90^\circ$. Thus $\triangle BAD$ is a right triangle.

    Now, note that $\angle CAB = 90^\circ - \angle ABC = \angle ADC$, so $\triangle ADC$ is similar to $\triangle BAC$. Thus,

    $$\frac{CD}{CA} = \frac{AC}{BC},$$

    and thus $CD = \frac{AC^2}{BC} = \boxed{\frac{16}{9}}$.

14. Define a sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 = 1$ and

    $$a_i = 81a_{i-1} + i$$

    for all integers $i \geq 2$. What is the smallest positive integer n such that a_n is divisible by 20?

    Answer: 15

    Solution: $81 \equiv 1 \pmod{20}$, so $a_i \equiv a_{i-1} + i \pmod{20}$. We just need $1 + 2 + \cdots + n = \frac{n(n-1)}{2}$ to be divisible by 20. The smallest such n is $\boxed{15}$ because we need a factor of 5 and at least 3 factors of 2 in the numerator.

15. Nikki chooses three distinct square cells, A, B, and C, from a 3×3 square grid uniformly at random. What is the probability that square C is contained within the rectangle whose opposite corners are squares A and B? The answer may be expressed in any form. Examples of rectangles with opposite corners at A and B are shown as shaded regions below.

Answer: $\mathbf{\frac{17}{63}}$

Solution: We proceed by casework, dividing the cases depending on the rectangle that's formed by the first two squares.

There are 12 possible configurations for a 1×2 block, and the probability the third square is outside this rectangle is 1.

There are 8 possible configurations for a 2×2 block, and the probability the third square is outside this rectangle is $\frac{5}{7}$.

There are 6 possible configurations for a 1×3 block, and the probability the third square is outside this rectangle is $\frac{6}{7}$.

There are 8 possible configurations for a 2×3 block, and the probability the third square is outside this rectangle is $\frac{3}{7}$.

Finally, there are 2 possible configurations for a 3×3 block, and the probability the third square is outside this rectangle is 0.

Combining all the cases gives us $\frac{1}{36}(12 \cdot 1 + 8 \cdot \frac{5}{7} + 6 \cdot \frac{6}{7} + 8 \cdot \frac{3}{7} + 2 \cdot 0) = \frac{46}{63}$. This is the probability that C is outside the rectangle defined by A and B, so the probability that C is inside that rectangle is $1 - \frac{46}{63} = \boxed{\frac{17}{63}}$.

16. Let a, b, and c be positive integers such that $\{a, b, c, 2025^2\}$ is a geometric sequence that is strictly increasing, meaning $a < b < c < 2025^2$. Find the number of distinct possible values of a.

Answer: 44

Solution: Denote the common ratio of the geometric sequence as r. Then, we can express the geometric sequence as a, ar, ar^2, ar^3. Therefore, $2025^2 = ar^3$. Notice that the prime factorization of $2025^2 = 3^8 \cdot 5^4$. Notice that since r^3 must be a perfect cube, the exponents in its prime factorization must all be multiples of 3. The maximum value of r^3 is therefore $3^6 \cdot 5^3$, so the smallest value of a is $3^2 \cdot 5^1$. All other possible values are multiples of this, and specifically it must be a perfect cube multiplied by this quantity, so a can be expressed in the form $a = 3^2 \cdot 5^1 \cdot k^3$. Since $k = 45$ gives $a = 2025^2$, k can only range from 1 to 44, so a can take on $\boxed{44}$ values.

17. Let (a, b, c, d, e) be a permutation of $(2, 3, 4, 5, 6)$. For example, a possible permutation is $(a, b, c, d, e) = (5, 3, 6, 2, 4)$. What is the maximum possible value of $a - ab + abc - abcd + abcde$, where ab, abc, $abcd$, and $abcde$ all represent multiplication (e.g., $ab = a \cdot b$), not permutations?

Answer: 652

Solution: Note that $abcde = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$, so this problem is asking us to maximize $a - ab + abc - abcd + 720$. Equivalently, it suffices to maximize $a - ab + abc - abcd$.

In this expression, $abcd$ is the most significant term, and it since it is subtracted it should be as small as possible. To make $abcd$ as small as possible, set $e = 6$, then $abcd = 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Then we are maximizing $a - ab + abc - 120$, so it suffices to maximize $a - ab + abc$ subject to $e = 6$.

In this expression, abc is the most significant term. Since it is added, it should be as large as possible. To make abc as large as possible, set $d = 2$, then $abc = 3 \cdot 4 \cdot 5 = 60$. Then we are maximizing $a - ab + 60$, so it suffices to maximize $a - ab$ subject to $e = 6, d = 2$.

We can continue this line of reasoning to find that the optimal values of a, b, and c are $4, 3$, and 5, respectively. Then the largest possible value of the original expression is $4 - 4 \cdot 3 + 4 \cdot 3 \cdot 5 - 4 \cdot 3 \cdot 5 \cdot 2 + 4 \cdot 3 \cdot 5 \cdot 2 \cdot 6 = \boxed{652}$.

So far, we have not proved that this *greedy* strategy produces the optimal value, but some experimentation with the problem should make you pretty convinced that this is the best we can do! For completeness, a more rigorous argument that this is the optimal value is given below.

We are maximizing $f(a, b, c, d, e) = a - ab + abc - abcd$. Suppose $c < e$. Then $f(a, b, c, d, e) > f(a, b, e, d, c)$ since $a - ab + abc - abcd > a - ab + abe - abed$. Thus, in the optimal permutation (a, b, c, d, e), we should have $c < e$, otherwise we could increase the value of f by switching c and e.

Now suppose $a < c$. Then $f(a, b, c, d, e) > f(c, b, a, d, e)$ since $a - ab + abc - abcd > c - cb + abc - abcd$. Thus, in the optimal permutation (a, b, c, d, e), we should have $a < c$, otherwise we could increase the value of f by switching a and c.

Now suppose $b < a$. Then $f(a, b, c, d, e) > f(b, a, c, d, e)$ since $a - ab + abc - abcd > b - ab + abc - abcd$. Thus, in the optimal permutation (a, b, c, d, e), we should have $b < a$, otherwise we could increase the value of f by switching a and b.

Now suppose $d < b$. Then $f(a, b, c, d, e) > f(a, d, c, b, e)$ since $a - ab + abc - abcd > a - ad + adc - abcd$. Thus, in the optimal permutation (a, b, c, d, e), we should have $d < b$, otherwise we could increase the value of f by switching b and d.

We have shown that the optimal permutation (a, b, c, d, e) must satisfy $d < b < a < c < e$, so the optimal permutation is $(a, b, c, d, e) = (4, 3, 5, 2, 6)$, as above.

18. Let $ABCDEF$ be a regular hexagon. Let P be a point on segment $\overline{BF}$, and let line $\overleftrightarrow{CP}$ intersect segment $\overline{AF}$ at Q. If $AB = 12$ and $\frac{BP}{BF} = \frac{3}{4}$, what is PQ?

Answer: $\mathbf{\frac{3\sqrt{43}}{7}}$

Solution: Extend $\overline{AF}$ and $\overline{CE}$ to meet at R. The key observation is that triangles $\triangle QPF$ and $\triangle QCR$ are similar. By standard 30-60-90 triangle ratios, we find that $BF = CE = 12\sqrt{3}$. Next, note that triangles $\triangle BEC$ and $\triangle FRE$ are congruent, so $ER = CE = 12\sqrt{3}$.

We quickly find $BP = 9\sqrt{3}$, and by Pythagoras $CP = 3\sqrt{43}$. Note that $PF = 3\sqrt{3}$, and furthermore,

$$\frac{PQ}{PQ + 3\sqrt{43}} = \frac{PQ}{CQ} = \frac{PF}{CR} = \frac{1}{8}.$$

Solving for PQ yields the answer $\boxed{\frac{3\sqrt{43}}{7}}$.

19. Danielle calculates the ones digit of each of the 2025 integers $1^{1!}, 2^{2!}, 3^{3!}, \cdots, 2025^{2025!}$. If Danielle adds up all these ones digits, what is the resulting value?

Answer: 6691

Solution: Notice that, for every term, we do only care about the units digit of the base. Now, let's consider unit digits.

- For any odd multiple of 5, x, and any positive integer n, x^{4n} ends in 5.
- For any even multiple of 5, x, and any positive integer n, x^{4n} ends in 0.
- For any odd number, x, that does not end in 5 and any positive integer n, x^{4n} ends in 1.
- For any even number, x, that does not end in 0 and any positive integer n, x^{4n} ends in 6.

Now, notice that $n!$ is divisible by 4 for all $n \geq 4$. Therefore, we count the number of values between 4 and 2025 inclusive in each of the above categories:

- All even multiples of 5 are multiples of 10, so there are $\lfloor \frac{2025}{10} \rfloor = 202$ of them (because neither $1, 2$, or 3 are divisible by 5).
- All other multiples of 5 are odd multiple of 5, so there are $\lfloor \frac{2025}{5} \rfloor - 202 = 203$ of them.
- There are an equal number of even and odd numbers, and $(2025 - 4 + 1) - 405 = 1617$ numbers between 4 and 2022 that are not multiples of 5. Since there are more odd multiples of 5 than even, we then have 809 even non-multiples of 5 and 808 odd non-multiples of 5.

Thus, the answer is $1 + 4 + 9 + (202 \cdot 0) + (203 \cdot 5) + (809 \cdot 6) + (808 \cdot 1) = \boxed{6691}$.

20. There exist strictly increasing arithmetic sequences of real numbers, $\{a, b, c\}$ and $\{p, q, r\}$, having the properties that q is a positive integer greater than 1 and that the equation $x^3 - ax^2 + bx - c = 0$ has solutions p, q, and r. Over all such pairs of increasing arithmetic sequences, what is the least possible value of $p + q + r$?

Answer: 18

Solution: Let $p = q - n$ and $r = q + n$ for some positive number n. Using Vieta's relations, we can generate the following equations:

$$a = (q - n) + q + (q + n) = 3q$$

$$b = (q - n)(q) + (q - n)(q + n) + q(q + n) = 3q^2 - n^2$$

$$c = (q - n)(q)(q + n) = q^3 - qn^2$$

Furthermore, $a + c = 2b$ since $\{a, b, c\}$ is an arithmetic sequence. Substituting our above expressions for a, b, c, we generate

$$q^3 - qn^2 + 3q = 6q^2 - 2n^2 \rightarrow q(q^2 - 6q + 3) = n^2(q - 2)$$

Since $n^2 > 0$, it must also be true that $\frac{q(q^2-6q+3)}{q-2} > 0$. Note that we don't have to worry about $q = 2$ causing a division by 0, as $q(q^2 - 6q + 3) = n^2(q - 2)$ fails for $q = 2$, so we can multiply both sides by $(q - 2)$ and divide both sides by q to yield $q^2 - 6q + 3 = (q - 3)^2 - 6 > 0$. Thus, $(q - 3)^2 > 6$ and $q \geq 6$. The sum $p + q + r$ is $a = 3q \geq \boxed{18}$.