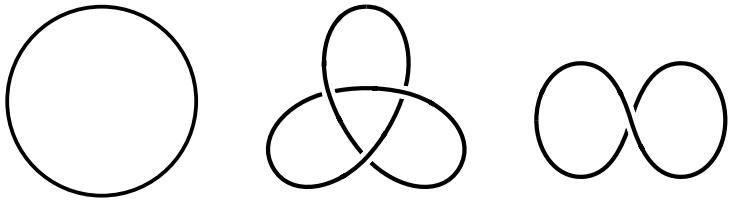
1 Introduction to Knots (31 pts)

Welcome to the BMT 2025 Power Round! In this round, we will explore the mathematics of knots. Understanding knots is useful for solving interesting problems in many fields, from protein folding to robotics. We will only have time to scratch the surface of this rich field, and we encourage you to explore it more on your own later.

In our study of knots, the ends are connected to each other, so that each knot is a closed loop. You can think of a **knot** as the result of taking a physical string, tying it up as you want, and gluing the two loose ends together. A **knot diagram** (or **knot projection**), by contrast, is a two-dimensional drawing of a knot, recording what parts of the string cross over or under other parts. A *crossing* is where one part of the string passes over another part. In no part of the diagram may more than two lines cross at the same point. See below for examples of knot diagrams. Make sure you understand the distinction between knots and knot diagrams! They are not interchangeable.

The simplest possible knot is the *unknot*. A knot is *trivial* if it can be deformed into a circle without cutting the string or allowing one part of the string to pass through another (see Question 1.5 for an example). A diagram of a trivial knot may still have crossings, as shown below. A circle with no crossings is one diagram of the unknot, but there are many others. When we refer to the zero-crossing diagram of the unknot, we mean this circle diagram. For the first part of this chapter, you may think of "transforming" a knot as continuously moving the closed loop without cutting it or allowing strands to pass through one another. Later, we will describe such transformations using Reidemeister moves.

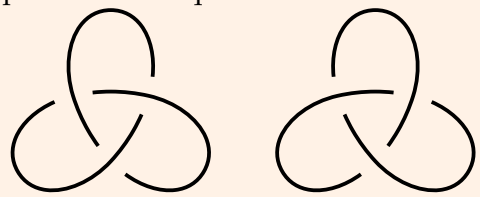
Another iconic knot is the trefoil knot, also shown below.



The unknot. A trefoil. A trivial knot.

Question 1.1. (2 pts) Construct a trefoil knot and an unknot out of the provided pipe cleaners. Without breaking the loop, is it possible to transform the trefoil into the unknot? If it is possible, draw a sequence of diagrams demonstrating the transformation. If it is not possible, no explanation is required.
Solution: If you construct the knots and play around, you should see that you cannot transform one into the other without breaking the loop. The answer is no.

Question 1.2. (3 pts) Are the left-trefoil knot diagram and the right-trefoil knot diagram (shown below) two diagrams of the same knot? That is, is it possible to transform one into the other without breaking the loops? If it is possible, draw a sequence of diagrams demonstrating the transformation. If it is not possible, no explanation is required.



Left-trefoil knot Right-trefoil knot

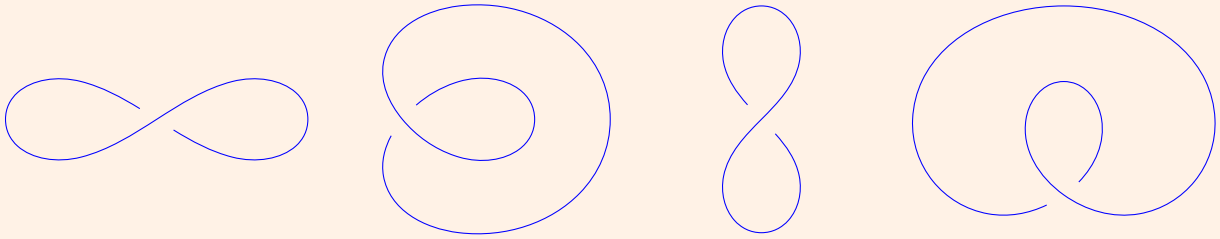
Solution: If you construct the knots and play around, you should see that you cannot transform one into the other without breaking the loop. The answer is no.

The above question illustrates an important topic in knot theory: equivalence.

How can we tell whether knots are equivalent? It turns out that there is not a simple answer. At this point in the round, we haven't yet developed methods to show that a given knot is nontrivial (it cannot be transformed into the unknot). To build up to this, we will first show that if a knot is nontrivial, then it must be at least as "complex" as a trefoil knot.

Question 1.3. (5 pts) Briefly explain why if a knot diagram has one crossing, the corresponding knot can be transformed into the unknot. (Hint: Draw the crossing first, and then make knot diagrams by connecting the ends in all possible ways so that no more crossings are created. Approach this task systematically so that your argument shows, without a doubt, that all possible knots have been considered.)

Solution: One-crossing diagrams (up to reflection and moving strands without creating crossings) are shown below. Each of them can be transformed into the unknot by some untwisting.



To show these are the only possibilities, label the four ends around the crossing in order. The four ends must be paired up without intersecting, and this is possible as long as 1 is not paired with 3, or 2 with 4.

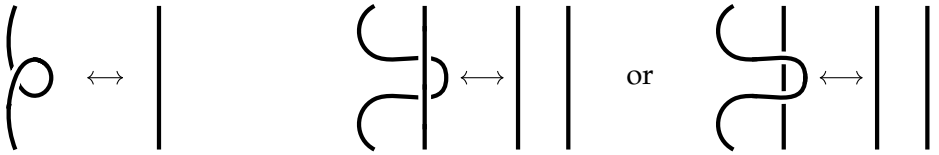
In order to show that a nontrivial knot is at least as complex as a trefoil knot, we need to show the same property holds for knots with two crossings: all of them can be transformed into the trivial knot. This can be shown with some more casework.

Thus, the simplest nontrivial knot must have at least 3 crossings. The trefoil knot has 3 crossings, but we haven't yet shown whether it is equivalent to the trivial knot. Proving that knots are nontrivial will take some extra machinery, which we'll develop more in Chapter 2.

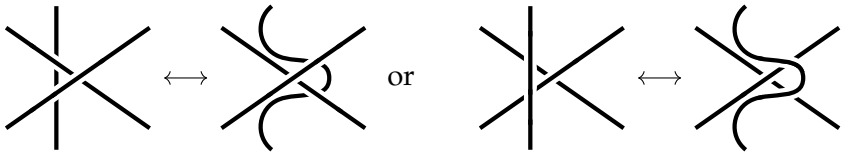
Now, let's return to an idea we've used in all of the previous problems: "transformations" of a knot. We want to make this idea more rigorous, so we can think about what happens to a knot under some transformation.

From now on, we'll define a transformation as a sequence of Reidemeister moves, which are ways to alter a knot diagram while preserving the knot it represents. Ordinary planar deformations of the diagram are also allowed. When making a Reidemeister move, the diagram remains unchanged except for the change indicated by the move. There are three Reidemeister moves, as defined below.

Definition 1.1. The **first Reidemeister move** is a simple twist or untwist of the knot.
The **second Reidemeister move** represents pulling a strand of the knot under or over another segment of string, creating or deleting two consecutive crossings.
The **third Reidemeister move** represents pulling a strand of the knot under or over another crossing. The three crossings retain their over/under relations, and the total number of crossings stays the same.



First Reidemeister move (R1) Second Reidemeister move (R2)



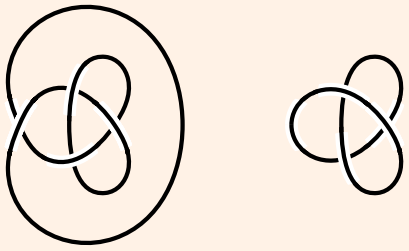
Third Reidemeister move (R3)

These moves might seem random, but it turns out that any valid transformation we make is actually a sequence of these Reidemeister moves! Even more importantly, the following theorem holds.

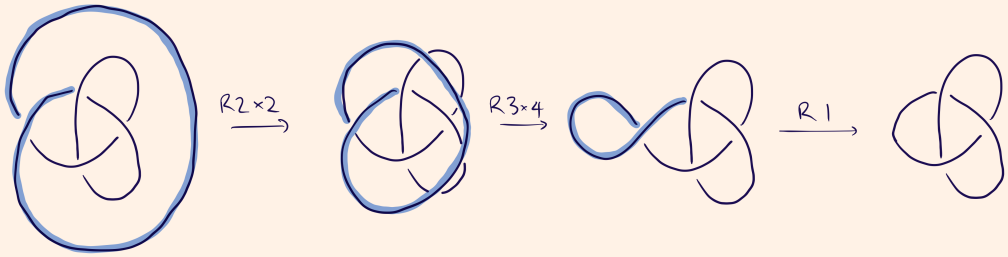
Theorem 1.2. (Reidemeister's Theorem): For any two diagrams of the same knot, there exists a sequence of Reidemeister moves that transforms one diagram into the other.

We won't prove this theorem, but its claim is incredible! We have three very simple moves we can perform on a knot diagram to get any other diagram of the same knot!

Question 1.4. (3 pts) Show that the two diagrams below represent the same knot by finding a sequence of Reidemeister moves from one to the other.

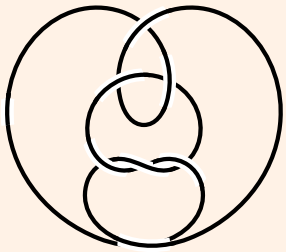


Solution:

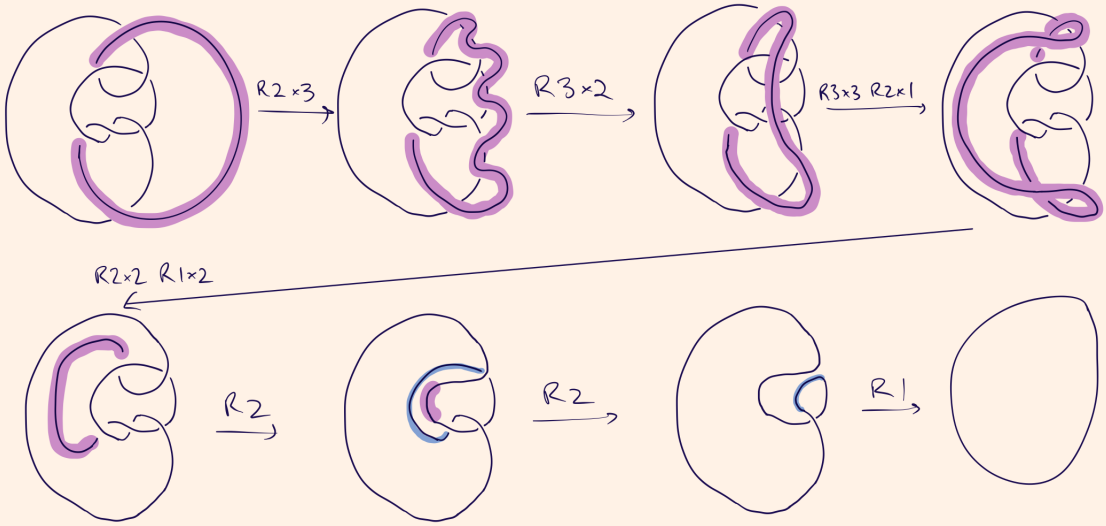


Between the second and third diagrams, the four R3 moves are followed by two R2 moves (not shown).

Question 1.5. (4 pts) The knot diagram shown below depicts the unknot. Find a sequence of Reidemeister moves that untangles it into the 0-crossing diagram shown before.

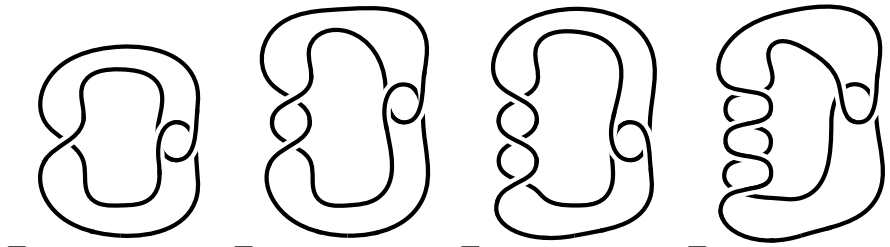


Solution:



We will return to Reidemeister moves in the next section, but for now let's play a game! In the following game, called the Knotting-Unknotting Game, we begin with a knot diagram and hide the crossings. This will be the board for the game. On each turn, a player chooses an unresolved crossing and resolves it by setting it to either over or under ($\times$ or $\times$). One player's goal is to create a nontrivial knot while the other player's goal is to create a trivial knot. Players alternate taking turns until all crossings are resolved. Player 1 goes first and Player 2 goes second.

Definition 1.3. A **twist knot** of n half-twists, denoted T_n, is obtained by twisting a loop n times and then connecting the ends together so that the knot is alternating, as shown below. T_n is nontrivial for all $n \geq 1$.



Question 1.6. (4 pts) For T_1, explain whether Player 1 or Player 2 has a winning strategy in the Knotting-Unknotting Game, assuming both players play optimally, if...

(a) Player 1's goal is to create the unknot.

(b) Player 1's goal is to create a nontrivial knot.

Solution: The unknotter always wins: (a) Player 1 wins, (b) Player 2 wins.

Let the "clasp" of the knot be the two crossings on the right. If the crossing types on the clasp are not alternating, the knot will definitely be the unknot. Therefore, if the unknotter finishes the clasp, they can guarantee a win. Here are winning strategies for the unknotter in both cases:

(a): P1 plays on the left side, then P2 starts the clasp. P1 finishes the clasp and so can make the knot trivial.

(b): P1 *must* play on the left side, otherwise they start the clasp and P2 can finish the clasp and win. Then, P2 resolves the inner clasp crossing to be the *opposite* of the crossing P1 chose. No matter what P1 chooses next, the resulting knot is the unknot; the inside part of the string can be pulled to remove both inner crossings.

Alternative argument: there are only two ways to resolve the crossings to produce a nontrivial knot: T_1 and T_1 with all crossings switched. After any first move, there is only one choice for each of the next two crossings that creates a nontrivial knot, so the unknotter can always make a choice that results in the unknot.

Question 1.7. (10 pts) Explain who has the winning strategy in the Knotting-Unknotting Game played on T_n, which could depend on Player 1's goal and the value of n.
Solution: Case 1: n is even.

Let the "clasp" of the knot be the two crossings on the right. Recall that if the unknotter finishes the clasp, they can guarantee a win.

First, we show P2 can always win on T_2. Suppose P1 wants a nontrivial knot. If P1 plays on the clasp, P2 can immediately win by finishing the clasp and guaranteeing the unknot. If P1 plays on the left, P2 can continue to play on the left until P1 is forced to make the first move on the clasp, allowing P2 to make the unknot.

Suppose P1 wants a trivial knot. When P1 plays on the clasp, P2 can finish the clasp with a crossing of the opposite type, and when P1 plays on the left side, P2 can play the same crossing type on the other left crossing. Regardless of the order of these pairs of moves, the resulting knot is always nontrivial!

We proved the base case that P2 can win on T_2. We will prove that if P2 can win on T_{n-2} with $n-2 \geq 2$, P2 can win on T_n as well. This will allow us to conclude that P2 can always guarantee a win when n is even.
If P1's first move is on the left side, P2 can choose the opposite resolution at a crossing adjacent to the one P1 resolved on the left side, which will "unwind" to create a board of size T_{n-2} where P2 is still the second to move. By the induction hypothesis, P2 will win.

Otherwise, suppose P1's first move is on the clasp.

- If P1's goal is the unknot, P2 will finish the clasp so that it remains clasped. Then P2 can choose the opposite resolution at an adjacent crossing on the left side through the penultimate move, so all pairs of moves "unwind". At the last move, P2 plays the same move, and the game board remains T_2 at the end. P2 wins.

- If P1's goal is nontrivial, P2 will finish the clasp the opposite way so that it can be "unclasped" by R2. Then any moves on the left side result in a trivial knot. P2 wins.

This concludes the induction step. So P2 can win on a board of size n for all even $n > 2$.

Case 2: n is odd.

If P1's goal is to create the unknot, they win. There is an odd number of crossings, so P1 can force P2 to be the first to play a move on the clasp. Then P1 can finish the clasp and guarantee an unknot.

If P1's goal is a nontrivial knot, they lose. P2 can use the same strategy as for even $n > 2$. We know that for T_1, the player whose goal is the nontrivial knot always loses. For odd $n \geq 3$,

- If P1's first move is on the left, P2 can play the opposite move on an adjacent crossing to reduce the board to T_{n-2}.

- If P1's first move is on the clasp, P2 can finish the clasp and force a trivial knot, as previously discussed.

P2 wins in both of these subcases. Thus, if n is odd, the unknotter always has a winning strategy.

2 Invariants (35 pts)

So far, we've avoided the problem of determining whether two knots are equivalent, meaning that one can be transformed into the other using Reidemeister moves. Let's tackle this problem using invariants.

Generally, an invariant is a property that doesn't change under some type of operation. For example, the number of fingers on my hand is independent of the order in which I count them. We want to think about properties of knot diagrams that are invariant under Reidemeister moves since we noted before that all diagrams of a knot are equivalent up to these moves.

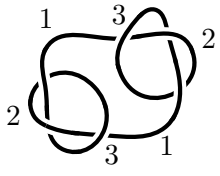
Definition 2.1. A **knot invariant** is a property that has the same value for every diagram of a given knot. Since any two diagrams of the same knot are related by Reidemeister moves and planar deformations, an equivalent definition is that the property is unchanged by these transformations.

Question 2.1. (3 pts) Is the number of crossings in the knot diagram of a given knot an invariant? That is, does the number of crossings of a knot diagram stay the same under Reidemeister moves? Prove that it does or give a counterexample.
Solution: This is not an invariant. Consider R1 (twisting a section of the knot), which adds or removes a crossing from a knot diagram. For a concrete counterexample, take a circle (no crossings) and apply R1 to create one crossing.

An important feature of knot invariants is that we can use them to show that two knots are *not* equivalent. If two knots have different values of an invariant, we definitely can't transform one into the other.

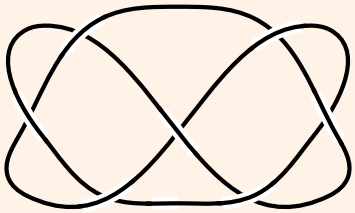
Definition 2.2. A **strand** in a knot diagram is a continuous line in the diagram that goes from one undercrossing (the string is below another part at a crossing) to another undercrossing, and any crossings in between are overcrossings (where the string passes above another part). In other words, a strand is a continuous visible arc, extended in both directions until it reaches an undercrossing. If there are no undercrossings, the entire closed curve is one strand.

Definition 2.3. A knot diagram is **tricolorable** if each of the strands in the diagram can be colored one of three different colors so that at least two colors are used in total and at each crossing, the three incident strands either have three different colors or all have the same color.
When demonstrating tricolorability, you are welcome to use colors if you have them, or mark the color of each strand with numbers as demonstrated below.

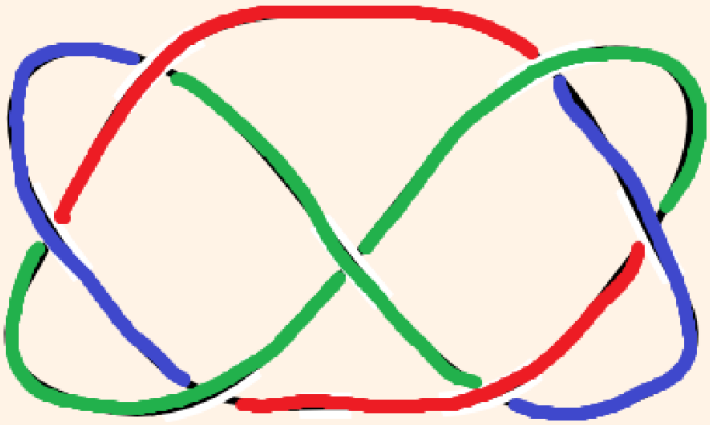


Question 2.2. (2 pts) Show that if we did not require at least two colors to be used, every knot would be tricolorable.
Solution: If using only one color were allowed, then any diagram could be tricolored by making all strands the same color.

Question 2.3. (3 pts) Show that the knot diagram below is tricolorable by coloring it.



Solution:

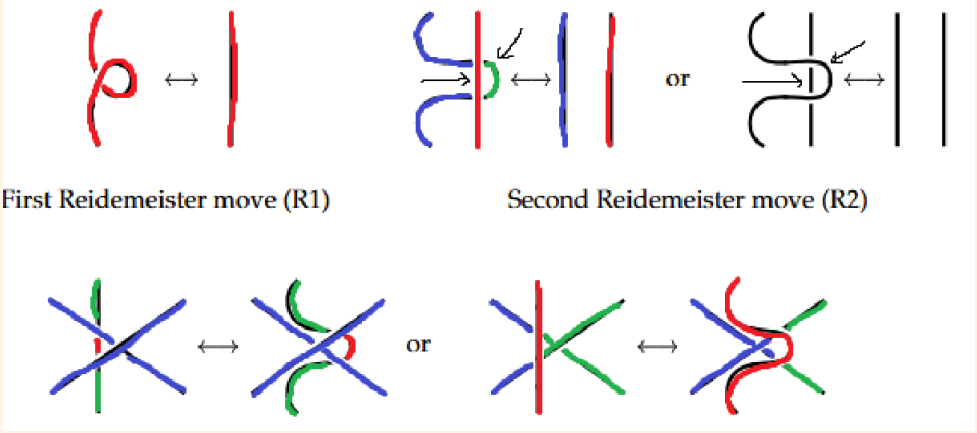


Theorem 1.2 from Chapter 1 tells us that tricolorability is a knot invariant if it is maintained under each of the three Reidemeister moves.

Question 2.4. (4 pts) Prove that tricolorability is a knot invariant by showing it is invariant under Reidemeister moves.

(a) Prove that tricolorability is invariant under type 1 Reidemeister moves.
(b) Prove that tricolorability is invariant under type 2 Reidemeister moves.
(c) Prove that tricolorability is invariant under type 3 Reidemeister moves.

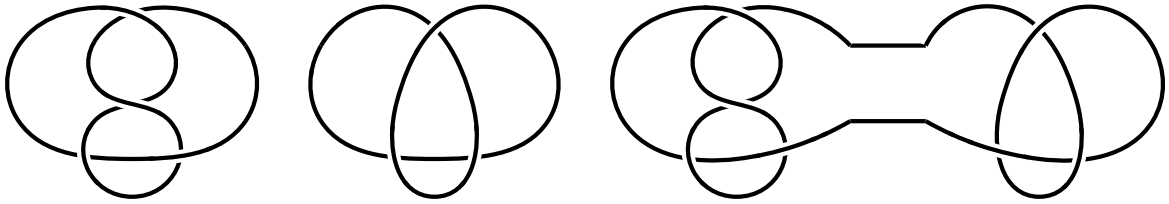
Solution: The diagrams below demonstrate that a knot is tricolorable before an R1, R2, or R3 move if and only if it is tricolorable after the move.



Question 2.5. (3 pts) Using tricolorability, prove that the trefoil knot is not equivalent to the unknot.
Solution: The trefoil is tricolorable, as shown below. The unknot is not tricolorable since it has only one strand and at least 2 colors must be used in a tricoloring. Since tricolorability is a knot invariant, the trefoil is not equivalent to the unknot!



Definition 2.4. Given two knot diagrams J and K, the **composition** $J\#K$ of two knot diagrams is obtained by removing a small exposed arc from each of two disjoint diagrams and joining the four endpoints with two new arcs that introduce no additional crossings, as shown below.



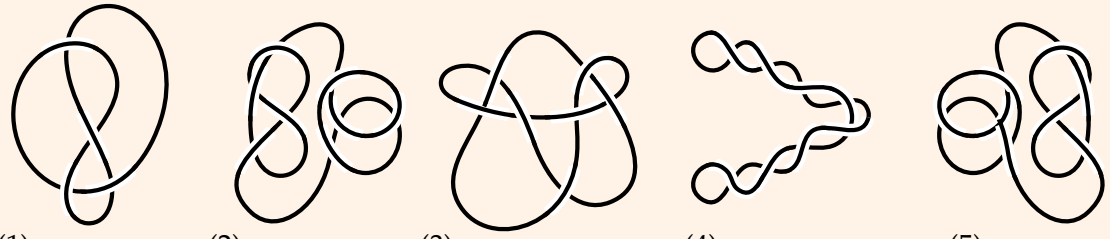
K_1 K_2 $K_1\#K_2$

Question 2.6. (2 pts) Find a knot I such that for all K, $K\#I = K$. The knot I is called the identity element for the operation $\#$.
Solution: The unknot.

Question 2.7. (3 pts) Prove that the composition of two tricolorable knots is also tricolorable.
Solution: Since both knots are tricolorable, we may choose a coloring for each such that the strands that will be connected are the same color. Then we can make the connection, and colors will be preserved at all crossings, so the final diagram is tricolorable.

Tricolorability doesn't necessarily tell us whether two knot diagrams are equivalent, but in some cases it tells us that two diagrams are *definitely not equivalent*: one is tricolorable and the other isn't.

Question 2.8. (4 pts) Exactly three of the following five diagrams represent the same knot; the remaining two are not equivalent to that knot. Determine which two are not equivalent to the other three.



(1) (2) (3) (4) (5)

Solution: First, observe that (1) and (2) are equivalent: (2) has two removable loops on the right, and once we use R2 and R1 to remove them, the knots are the same. Also, notice (1) is not tricolorable: it has 4 crossings, and each strand meets every other strand. From this we know (2) is not tricolorable either. (3) is tricolorable, so it cannot be equivalent to (1) and (2). (4) is equivalent to the unknot, so it is not equivalent to (1), (2), or (3). Thus, by the process of elimination, (3) and (4) are different, and (1), (2), and (5) are the same.

Now, let's look at another property of knot diagrams. If we pick a point on a knot diagram to start and then continuously trace the knot diagram in either direction (creating an oriented closed curve), each crossing will have a "direction" to it.

Definition 2.5. The **sign** of a crossing is 1 if the lower strand points right compared to the direction of the upper strand, and -1 if it points left.



Crossing with sign -1 Crossing with sign 1

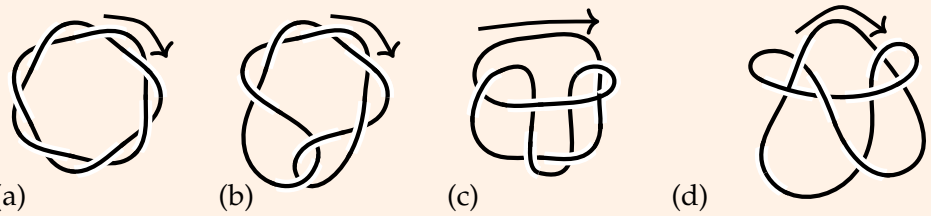
Definition 2.6. The **writhe** of a knot diagram is the sum of the signs of all crossings in the diagram.

Let's explore writhe to see how it is useful.

Question 2.9. (3 pts) Does the writhe of a diagram depend on which direction we trace the knot in? If so, how does it change? State and prove a claim.

Solution: Writhe does not depend on orientation. When we reverse orientation, both arrows change direction, so the understrand remains on the same side relative to the direction of the overstrand.

Question 2.10. (4 pts) Compute the writhe of the following knot diagrams. If you determined that writhe depends on orientation, follow the knot in the direction as marked.



(a) (b) (c) (d)

Solution:

- (a): 7
- (b): -7
- (c): -3
- (d): 1

Let's conclude the section by determining whether writhe is a knot invariant.

Question 2.11. (4 pts) For **each** of the three Reidemeister moves, prove or disprove the invariance of writhe. Then, conclude whether writhe is a knot invariant.

Solution:

- R1: Not invariant. R1 adds or removes one crossing of sign $+1$ or -1, so it changes writhe by 1 or -1.
- R2: Invariant. The two created/removed crossings must have opposite sign, so they sum to zero.
- R3: Invariant. Each of the three crossings keeps its sign: the same strands cross with the same directions and the same strand on top.

We conclude writhe is not a knot invariant.

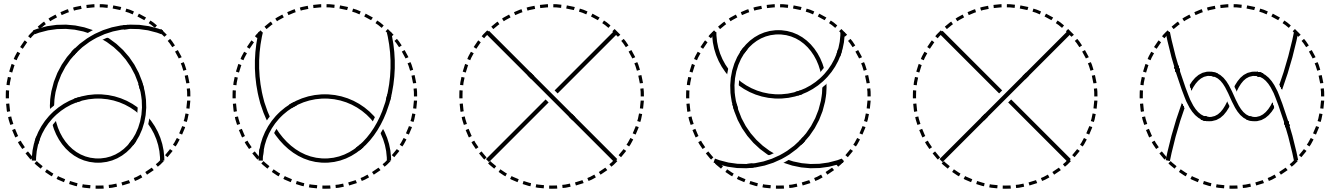
It turns out that writhe is very useful for the Kauffman polynomial, which gives rise to interesting invariants that we won't explore in depth in this Power Round. Furthermore, we can adjust the definition of writhe to find an invariant for a generalization of knots called virtual knots.

3 Rational Tangles (60 pts)

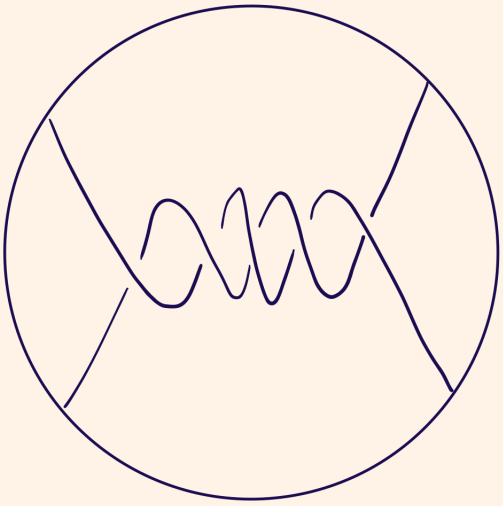
Let's now turn our focus to *rational tangles*. Tangles describe parts of knots, and rational tangles can be built using the operations introduced in this chapter.

Definition 3.1. A **tangle diagram** is a diagram contained in a disk, containing two strings with fixed endpoints on the boundary of the disk, possibly some closed components, and possibly crossings in the middle. For convenience, we place the fixed endpoints in the NW, NE, SE, and SW compass directions.

Here are some examples of tangle diagrams.



Question 3.1. (2 pts) Draw a tangle diagram with five crossings.
Solution: Example:

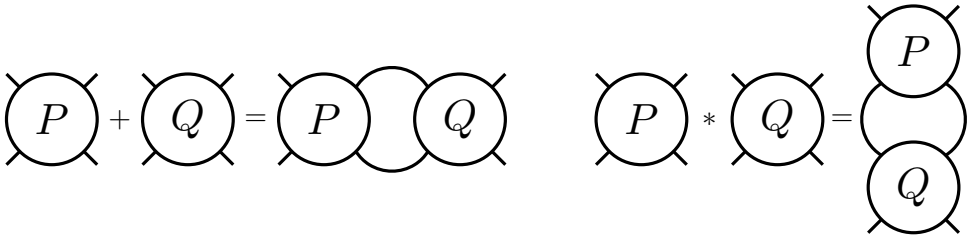


Definition 3.2. Two tangle diagrams P and Q are **equivalent** if one can be moved to look identical to the other using the Reidemeister moves provided in Chapter 1, while keeping the endpoints fixed. If P and Q are equivalent, we write $P \sim Q$.

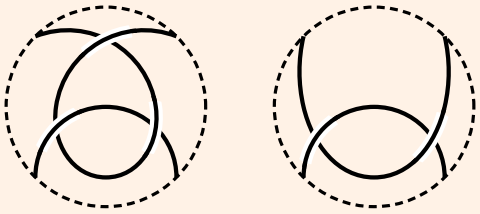
In order to analyze tangles, we first introduce some notation.

Definition 3.3. Given two tangles P and Q, we define:

- The **sum**, $P + Q$, is the result of horizontally connecting P and Q by connecting the top right endpoint of P with the top left endpoint of Q, and similarly for the bottom points.
- The **product**, $P * Q$, is the result of vertically connecting P and Q and connecting adjacent endpoints.
- The **negation**, $-P$, is the result of switching all crossings in P (from $\times$ to $\times$ and vice versa).
- The **inverse**, P^{-1} or $\frac{1}{P}$, is the result of rotating P by 90 degrees clockwise and switching all crossings (taking the negation).
- The **difference**, $P - Q$, is defined by $P - Q = P + (-Q)$.



Question 3.2. (4 pts) Let T_1 and T_2 be the following tangles.

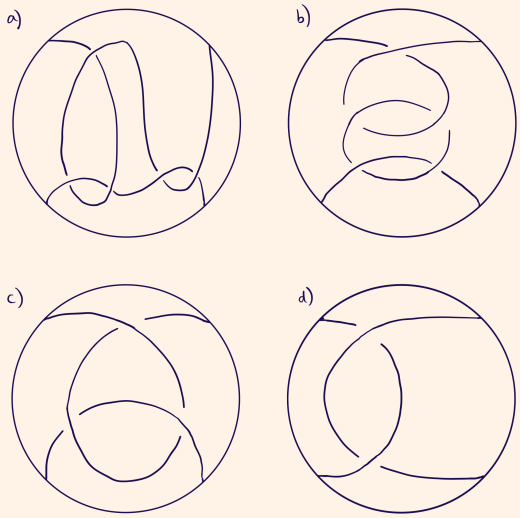


T_1 T_2

Draw:

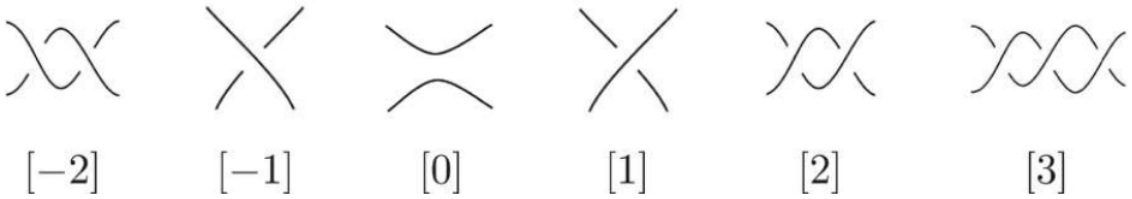
(a) $T_1 + T_2$
(b) $T_1 * T_2$
(c) $-T_1$
(d) $\frac{1}{T_2}$

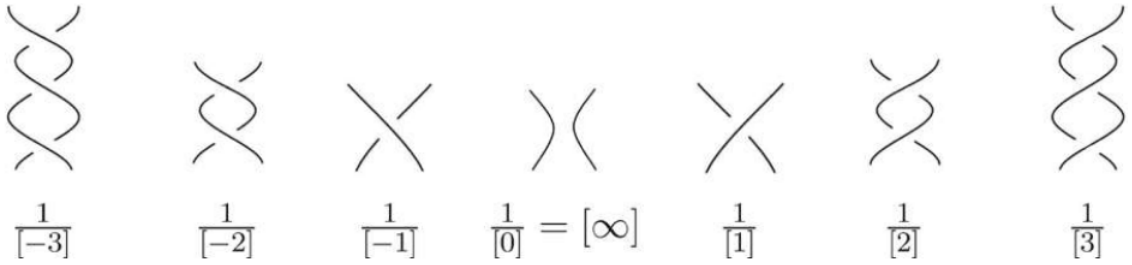
Solution:



- $T_1 + T_2$ has the two tangles connected horizontally.
- $T_1 * T_2$ has the two tangles connected vertically.
- $-T_1$ has every crossing switched.
- $1/T_2$ is obtained by rotating T_2 clockwise by $90°$ and switching every crossing.

Definition 3.4. For nonnegative integers N, the **integer tangle** $[N]$ is the tangle with N horizontal crossings, with the first crossing being of type $\times$ and subsequent crossings alternating type as shown below. We define $[-N] = -[N]$. A **reciprocal tangle** is the inverse $\frac{1}{[N]}$ of an integer tangle. We define $\frac{1}{[0]} = [\infty]$ and $\frac{1}{[\infty]} = [0]$.





Integer and reciprocal tangles are the building blocks for our exploration of rational tangles!

Definition 3.5. Rational tangles have no closed components. A **rational tangle** is any tangle you can make by starting with an integer or reciprocal tangle and then repeating a finite number of steps where you either add an integer tangle or multiply by a reciprocal tangle on the left or right.

For example, $[1] + \frac{1}{[2]}$ is a rational tangle created by adding the integer tangle $[1]$ to $\frac{1}{[2]}$ on the left.

Theorem 3.6. For every rational tangle P, $(P^{-1})^{-1} \sim P$. In other words, taking the inverse twice gives an equivalent tangle (the diagram itself has made a planar half-turn).

Note that in order to keep some nice properties of rational tangles, adding a reciprocal tangle to a rational tangle or multiplying a rational tangle by an integer tangle doesn't always result in a rational tangle.

Question 3.3. (5 pts) Prove the following statements about rational tangles.

(a) For all integers m and n, $[m] + [n] \sim [m + n]$.

(b) For all integers m and n, $\frac{1}{[m]} * \frac{1}{[n]} \sim \frac{1}{[m]+[n]}$.

Solution: The solution path is similar for both.

(a):

- If either of m, n is zero, the result is trivial.
- If $m, n > 0$, then $[m]+[n]$ is the horizontal concatenation of m horizontal alternating crossings with n horizontal alternating crossings, which is just $m+n$ horizontal alternating crossings = $[m + n]$.
- If exactly one of m and n is negative, then after concatenation there are $\min(|m|, |n|)$ crossing pairs that will "unwind" using Reidemeister move 2. We are left with $|m + n|$ unpaired crossings.
  4 cases:
  - $|m| > |n|, m < 0, n > 0$: We are left with the first $|m| - |n|$ crossings only, which are negative. Hence we get $[-(|m| - |n|)] = [m + n]$.
  - $|m| > |n|, m > 0, n < 0$: We are left with the first $|m| - |n|$ crossings only, which are positive. Hence we get $[m - |n|] = [m + n]$.
  - Same cases, $|n| > |m|$. The logic is the same, but with the last $|n| - |m|$ crossings.
- If $m = -n$, all crossings cancel and the result is $[0] = [m + n]$.
- If both $m, n < 0$, switch every crossing in the positive-positive case to obtain $[m] + [n] \sim [m + n]$.

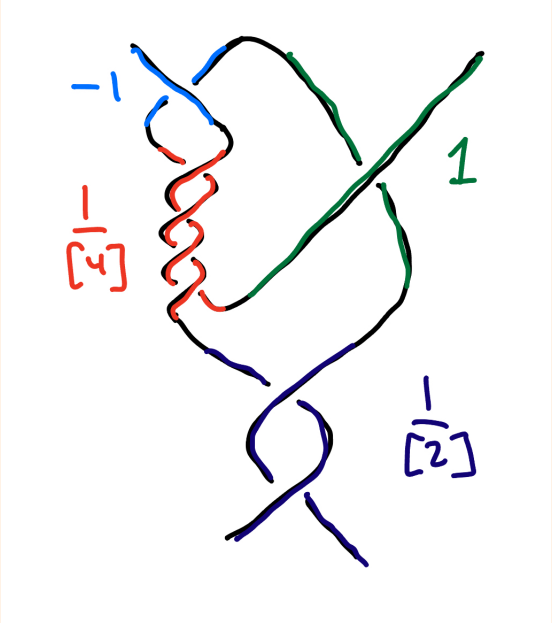
(b):

Notice $1/[m] * 1/[n]$ is the diagram of $[-m] + [-n]$, rotated $90°$ clockwise, and $1/([m] + [n])$ is the diagram of $[-(m + n)]$, rotated $90°$ clockwise. By (a), these two diagrams are equivalent.

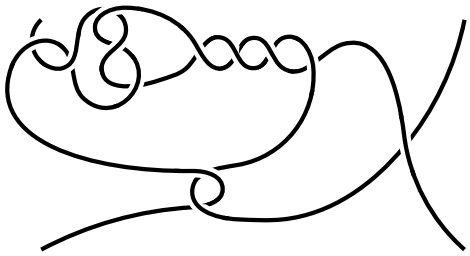
Question 3.4. (3 pts) Draw the following rational tangle:

$$\left(\left([-1] * \frac{1}{[4]}\right) + [1]\right) * \frac{1}{[2]}.$$

Solution:

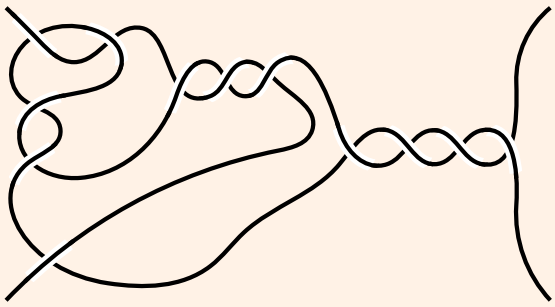


Sometimes, it is not immediately obvious how to construct a rational tangle from integer and reciprocal tangles, as shown in the example below.



is $\left(\left([2] + \frac{1}{[3]} + [-4]\right) * \frac{1}{[-2]}\right) + [-1]$.

Question 3.5. (4 pts) Find a construction for the rational tangle shown below from integer and reciprocal tangles.



Solution:

$$\left(\left(\left([-2] * \frac{1}{[2]}\right) + [3]\right) * \frac{1}{[1]}\right) + [-4]$$

Question 3.6. (7 pts) For a rational tangle P, we define P^h to be the horizontal flip of P, obtained by rotating the tangle diagram around the horizontal axis $180°$. Similarly, we define P^v to be the vertical flip of P, obtained by rotating the diagram through $180°$ about the vertical axis. Prove that $P \sim P^h$ and $P \sim P^v$. (Hint: use induction.)

Solution: A key idea is that every rational tangle is built from the tangles $[0], [\infty], [1], [-1], \frac{1}{[1]}, \frac{1}{[-1]}$ by Question 3.3. We use these tangles as our base cases. It's also possible, though more difficult, to have integer and reciprocal tangles as our base cases.

It is straightforward to see that these tangles are equivalent to their horizontal and vertical flips—the upper crossing becomes the lower crossing, and the positions of the two strands are mirrored, restoring the original crossing.

Now, the inductive step. If the desired property holds for a tangle diagram (call it T_n) composed of n of these "base" tangles, then:

$$(T_n + [\pm 1])^h \sim T_n^h + [\pm 1]^h \sim T_n + [\pm 1]$$

$$(T_n + [\pm 1])^v \sim [\pm 1]^v + T_n^v \sim [\pm 1] + T_n \sim T_n + [\pm 1].$$

The last equivalence holds because a twist $[\pm 1]$ on the left can be moved to the right by "untwisting" it—flipping T_n horizontally and adding the twist $[\pm 1]$ on the right; this is easier to see with a diagram.

For products, the vertical flip does not reorder the factors:

$$\left(T_n * \frac{1}{[\pm 1]}\right)^v \sim T_n^v * \left(\frac{1}{[\pm 1]}\right)^v \sim T_n * \frac{1}{[\pm 1]}.$$

For the h-flip, which does reorder the factors, use the same logic as in the second addition case, with the diagram turned a quarter turn.

Let's use our new notation to make some conclusions about when two tangles are equivalent.

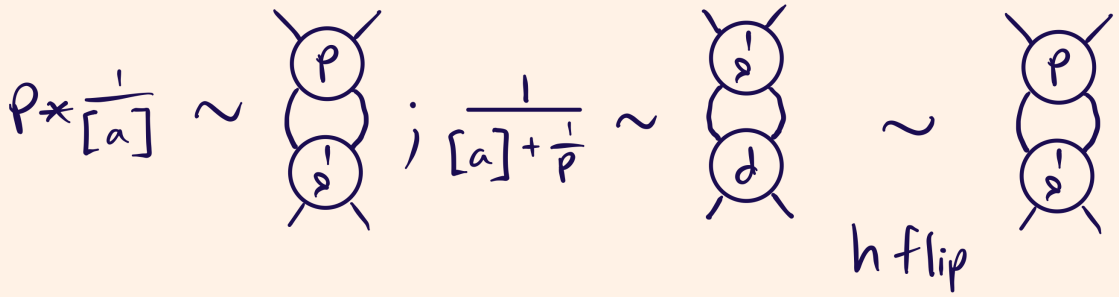
Theorem 3.7. (Product-to-Inverse Equivalence): Given a rational tangle P and integer a,

$$P * \frac{1}{[a]} \sim \frac{1}{[a] + \frac{1}{P}}, \qquad \frac{1}{[a]} * P \sim \frac{1}{\frac{1}{P} + [a]}.$$

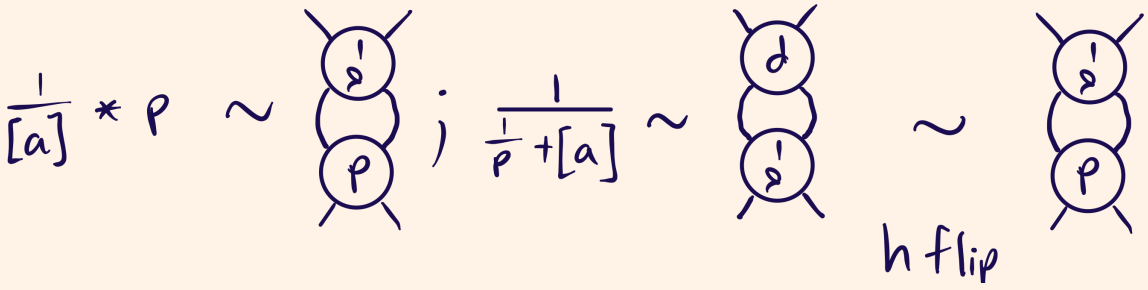
While this might seem at first to be an unnecessary observation, it turns out that being able to represent rational tangles in a form that looks like a continued fraction can be very useful. This theorem allows us to change an arbitrary rational tangle construction into a form that looks more like a continued fraction.

Question 3.7. (5 pts) Prove the Product-to-Inverse Equivalence by demonstrating an appropriate sequence of Reidemeister moves and diagrams to transform one side into the other.

Solution:



By Question 3.6, $d \sim d^h \sim P$; $\sim$



In the last couple of chapters, we've spent a lot of time trying to figure out when two knots are equivalent. Using our new knowledge from this section, we can make a significant conclusion about when two rational tangles are equivalent.

First, our constructions of tangle diagrams like in Question 3.5 didn't necessarily look anything like a continued fraction. Let's find a standardized way of representing these tangles.

Definition 3.8. The **continued fraction form** of a rational tangle is a construction of a rational tangle using a continued fraction of integer tangles $[a_1], \ldots, [a_n]$ with $a_1 \neq 0$, as shown:

$$[a_n] + \cfrac{1}{[a_{n-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}}.$$

The **associated fraction** of the tangle is

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

We also allow the crossing-free tangles $[0]$ and $[\infty]$ as exceptional continued fraction forms. Fractions are evaluated in $\mathbb{Q} \cup \{\infty\}$, with $1/0 = \infty$, $1/\infty = 0$, and $a + \infty = \infty$ for finite a.

Before we can use this form, we need to show that every rational tangle admits a continued fraction form. Arbitrary coefficient lists need not be unique. To see this, we can use the following theorem, which will also be very helpful in solving later problems in this chapter. Think about what it is saying for a second; it is a remarkable result that we unfortunately don't have time to prove during this round.

Theorem 3.9. For any rational tangle P and integer m,

$$P + [m] \sim [m] + P,$$

$$P * \frac{1}{[m]} \sim \frac{1}{[m]} * P.$$

Question 3.8. (6 pts) Given a construction of a rational tangle (a sequence of integer and reciprocal tangles combined using $+$ and $*$, with parentheses to group terms), describe an algorithm to "translate" this construction into continued fraction form. (Hint: Use Product-to-Inverse Equivalence.)
Solution: Consider the outermost term in the construction (outside the grouping parentheses). If it is not on the left, move it to the left using Theorem 3.9. If it uses a $+$ operation, combine it with any existing integer tangle on the left if applicable, then continue. If it uses a $*$ operation, the construction must have the form $\frac{1}{[a]} * X$, and we apply the Product-to-Inverse Equivalence to get $\frac{1}{\frac{1}{X}+[a]}$. Applying Theorem 3.9 again, this is equivalent to $\frac{1}{[a]+\frac{1}{X}}$. Repeat this process recursively until the remaining expression is an integer or reciprocal tangle.

The algorithm establishes that every rational tangle admits a continued fraction form. Question 3.9 proves uniqueness for normalized positive numerical expansions, and Question 3.11 handles general signs at the tangle level. Now that we know this is a valid representation of rational tangles, let's take our first look at our main theorem of this chapter!

Theorem 3.10. (Conway's Theorem): Two rational tangles are equivalent if and only if their associated fractions are equal.

This is a very powerful theorem, and we'll finish this chapter by working towards a proof of part of this statement. Let's start by thinking a bit more about continued fractions in general.

Question 3.9. (4 pts) Prove that for any fraction $\frac{p}{q} > 1$, there is a unique finite sequence of positive integers $a_1, \ldots, a_n$ such that $a_1 > 1$ and

$$\frac{p}{q} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

Solution: Let $x = p/q, b_0 = \lfloor x \rfloor$. Then $p/q = b_0 + r_0$, $0 \leq r_0 < 1$. If $r_0 = 0$, we are done. Otherwise, let $x_1 = 1/r_0$, and $b_1 = \lfloor x_1 \rfloor$, so $x_1 = b_1 + r_1$. Again, if $r_1 = 0$, done. Because x is rational, the remainders r_k are rational numbers whose denominators (in lowest terms) strictly decrease under the Euclidean algorithm, so this process must terminate after finitely many steps with $r_k = 0$ for some finite k. Every quantity whose integer part we take is greater than 1, so the last integer b_k is at least 2.

The sequence $a_1, \ldots, a_n$ corresponds to $b_k, \ldots, b_0$. At every step before the last one, the reciprocal term lies strictly between 0 and 1, so the outer term must be the integer part of the expression, guaranteeing uniqueness. Subtracting that coefficient and taking the reciprocal brings us to the next step. Once we reach a step with an integer value, the expansion must stop, since any additional reciprocal term would cause the fractional part of the number to be nonzero.

We also have the following formula:

$$a - \frac{1}{b} = (a-1) + \frac{1}{1 + \frac{1}{b-1}}$$

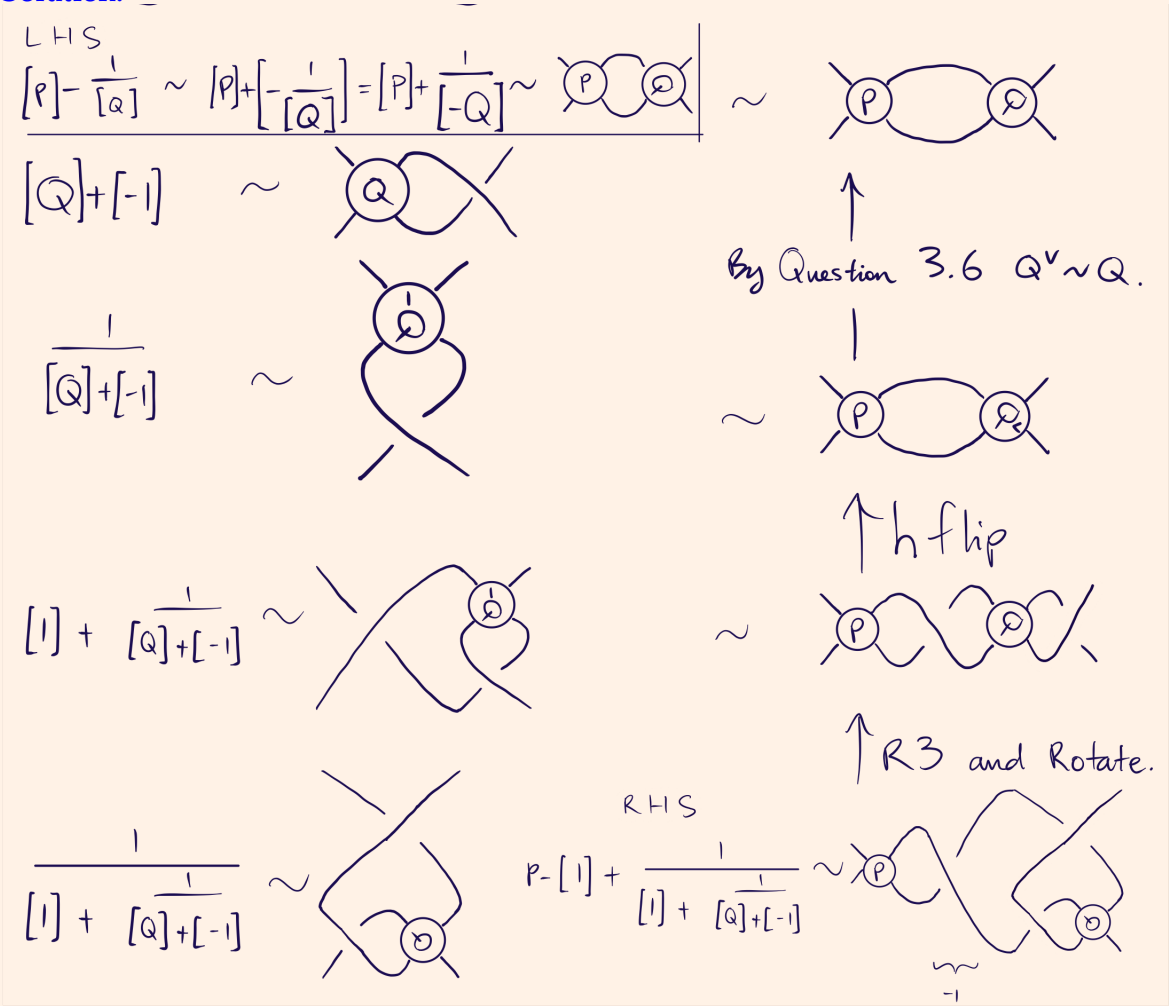
Amazingly, this formula is not just true for continued fractions, but also rational tangles!

Question 3.10. (12 pts) Prove that for any rational tangles P and Q,

$$P - \frac{1}{Q} \sim P - [1] + \frac{1}{[1] + \frac{1}{Q-[1]}}$$

by demonstrating an appropriate sequence of diagrams and/or Reidemeister moves to transform one side into the other.

Solution:



Now that we know this, we can prove the reverse direction of Conway's Theorem!

Question 3.11. (8 pts) Prove that if two rational tangles have equal associated fractions, then the tangles are equivalent. Remember that equal fractions need not have identical continued fraction forms, since the integer coefficients are not required to be nonnegative.

Solution: Assume P and Q are two rational tangles with the same associated fraction p/q. Let

$$P = [b_n] + \cfrac{1}{[b_{n-1}] + \cfrac{1}{\ddots + \cfrac{1}{[b_2] + \cfrac{1}{[b_1]}}}}.$$

$$Q = [c_m] + \cfrac{1}{[c_{m-1}] + \cfrac{1}{\ddots + \cfrac{1}{[c_2] + \cfrac{1}{[c_1]}}}}.$$

$$\frac{p}{q} = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{\ddots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}.$$

Since the coefficients b_i and c_i are not assumed to be positive, we cannot say that $a_i = b_i = c_i$ for all i (or even that the lists have the same length). Instead, consider a "canonical" rational tangle R, where

$$R = [a_k] + \cfrac{1}{[a_{k-1}] + \cfrac{1}{\ddots + \cfrac{1}{[a_2] + \cfrac{1}{[a_1]}}}}.$$

By using the formula $a - \frac{1}{b} = (a-1) + \frac{1}{1+\frac{1}{b-1}}$ repeatedly, we show we can convert any continued fraction into one with positive coefficients. Since Question 3.10 shows this formula also holds for tangles, showing this will be enough to conclude that $P \sim R$ and $Q \sim R$, so $P \sim Q$.

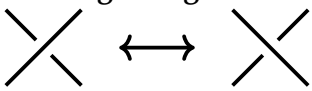
First, lift any negative coefficients to the numerator by multiplying the appropriate numerator and denominator by -1 (for tangles, this means switching all crossings in the appropriate sub-tangle). After this step, all coefficients will be positive except possibly the outermost coefficient, but there may be occurrences of $a - \frac{1}{b}$ (as opposed to addition only). Now, find the innermost occurrence of a minus sign (the one with smallest index). If it does not exist, we are in canonical continued fraction form. Otherwise, apply the formula above to get $(a-1) + \frac{1}{1+\frac{1}{b-1}}$. Since the coefficients are positive, subtracting 1 will never yield a negative, and after possibly some simplification (removing zero coefficients as needed), we are left with a continued fraction with one fewer negative sign and all positive coefficients. We therefore repeat this process until all coefficients except possibly the outermost coefficient are positive and there are no subtractions. At this point, we are in canonical form. After merging a final coefficient 1 into the one before it, Question 3.9 shows the list is unique when $p/q > 1$; for $p/q \leq 1$, apply it to the reciprocal of the fractional part, and treat 0 and ∞ separately as in Definition 3.8.

Since the same process on rational tangles preserves equivalence (using Question 3.10, Theorem 3.9, and the double-inverse equivalence), we may use the same operations to show $P \sim R$ and $Q \sim R$, and we are done.

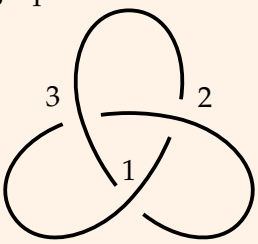
4 Unknotting Operations (79 pts)

As shown previously, many knot diagrams are equivalent to the unknot. More generally, we may take any knot, and by a series of operations, turn it into the unknot. One basic operation is the *crossing change*.

Definition 4.1. A **crossing change** is the operation of reversing which strand passes over the other at a single crossing. An example of a **crossing change** is shown below.

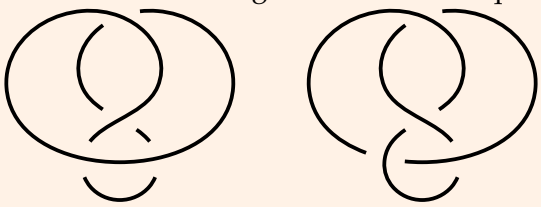


Question 4.1. (2 pts) Name a set of crossings in the knot diagram below that, when each changed, would result in the knot diagram being equivalent to the unknot.



Solution: Any set of one or two of the labeled crossings works. Changing no crossings leaves the original trefoil, and changing all three gives its mirror-image trefoil.

Question 4.2. (2 pts) Which of the two knot diagrams below are equivalent to the unknot?



Solution: Only the knot diagram on the left is equivalent to the unknot.

Question 4.3. (5 pts) Prove that in any knot diagram, there exists some set of crossing changes that can be made to make the diagram equivalent to the unknot.

Solution: Consider a knot in three-dimensional space whose projection from a certain direction has no crossings. This kind of knot must be equivalent to the unknot, even though it may have many crossings from other perspectives. One approach to making a knot projection equivalent to the unknot is to assume the projection is of the knot viewed from above, and resolve the crossings in a manner so that the knot has no crossings when viewed from the side.

Suppose that, as we traverse a knot in its chosen orientation, we always move downward. In this case, if the knot is viewed from the side, it has no crossings since points on the knot are at different heights. To ensure the knot is closed, we can assume the knot abruptly goes upwards at the end in a manner that does not create any crossings.

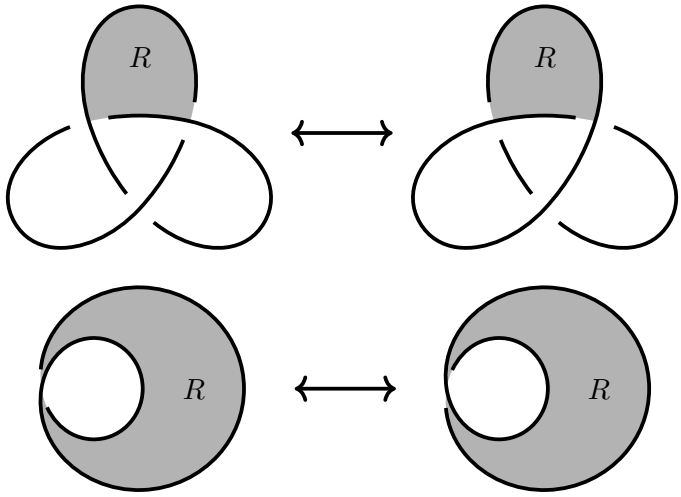
To achieve this from a knot projection, we resolve each crossing by having the strand pass over when we first reach the crossing. Since crossing changes can change how knot crossings are resolved, we can guarantee this resolution of crossings with crossing changes. Thus, crossing changes can unknot any knot diagram.

Question 4.4. (3 pts) Prove that in any knot diagram, the minimum number of crossing changes needed to make the diagram equivalent to the unknot is at most half of the total number of crossings in the diagram.
Solution: Let a knot diagram have N crossings. We have previously shown there must exist a subset with k of these crossings such that performing a crossing change on each of the k chosen crossings makes the diagram equivalent to the unknot. If $k \leq \frac{N}{2}$ we're done; otherwise, we can perform a crossing change on the other $N - k$ crossings, and we will still get a diagram equivalent to the unknot. This diagram is the mirror image of the unknot obtained using the first set: every crossing is reversed. Reflecting its unknotting moves shows that it is also the unknot. In this second case, $N - k < \frac{N}{2}$, so it is always possible to make the diagram equivalent to the unknot in at most $\frac{N}{2}$ crossing changes.

Since any knot can be converted into the unknot through a series of crossing changes, the crossing change is called an *unknotting operation*. Another operation on knots is the *region crossing change*.

Definition 4.2. A **region crossing change** (RCC) is an operation that performs a crossing change on each crossing on the boundary of a region in a knot diagram. Note that the region outside of a knot is considered a valid region for RCC moves.

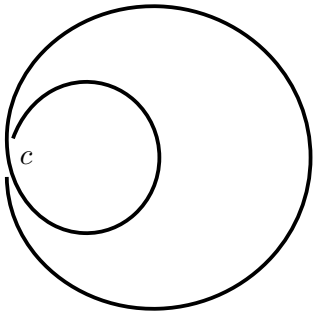
Examples of region crossing changes are shown below.



Question 4.5. (3 pts) Suppose that R and S are regions in a knot diagram. Prove that performing an RCC move on R followed by an RCC move on S produces the same modified knot diagram as performing an RCC move on S followed by an RCC move on R.
Solution: Each RCC changes a fixed set of crossings. A crossing is reversed if it is changed an odd number of times, and is unchanged if it is changed an even number of times. Reordering the RCC moves does not change these counts, so it does not change the result.

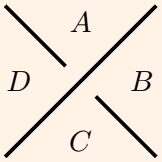
Notably, how RCCs affect the crossings of a diagram is related to how regions are positioned relative to crossings. In particular, crossings can be categorized by the regions around them.

Definition 4.3. A **reducible crossing** is a crossing where a region is adjacent to that crossing twice. For instance, crossing c is reducible in the knot diagram below. For the purposes of determining whether a crossing is reducible, the exterior of the knot is also considered a region.



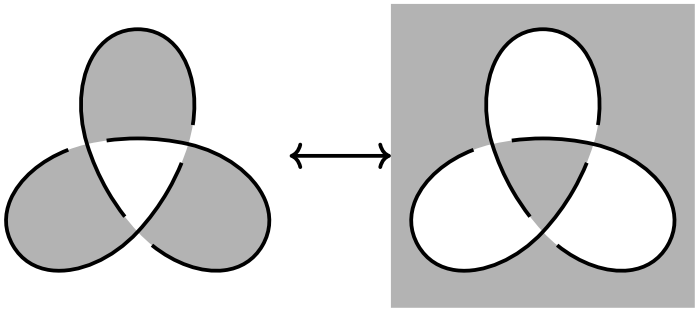
Definition 4.4. A **reduced diagram** is a knot diagram where none of the crossings are reducible.

Question 4.6. (2 pts) Suppose the crossing shown in the figure below is from an arbitrary knot diagram. Which regions (other than A) could be the same region as A?



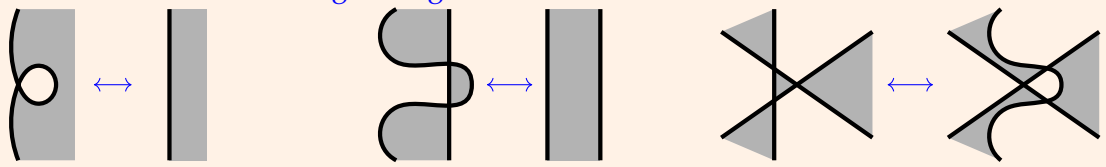
Solution: A can't be the same region as B or D since they are separated by the knot. A can be the same region as C; see the diagram illustrating a reducible crossing for an example. Thus, C is the only region that could be the same as A.

Definition 4.5. A **checkerboard coloring** of a knot diagram is the shading of certain regions in the diagram such that regions sharing an edge segment have opposite shades. Regions that meet only at a crossing are not considered adjacent. Two possible checkerboard colorings of a knot diagram are shown below. Note that the region outside the knot is also considered a region for the purposes of checkerboard coloring.



Question 4.7. (4 pts) Prove that checkerboard colorability is invariant under Reidemeister moves. In other words, if a Reidemeister move is performed somewhere on a checkerboard colorable knot diagram, the knot diagram can still be checkerboard colored. For this problem, you may not use the fact that all knot diagrams are checkerboard colorable, even if you are able to prove it independently.

Solution: A general approach that can be used to prove this is to start with some checkerboard coloring prior to performing the Reidemeister move and show there still exists a checkerboard coloring after the Reidemeister move is performed. In particular, we keep the coloring of the non-enclosed regions around our Reidemeister moves the same so that no other regions of the knot need their checkerboard coloring reassigned.



R1 R2 R3

Question 4.8. (5 pts) Prove that any knot diagram can be checkerboard colored.

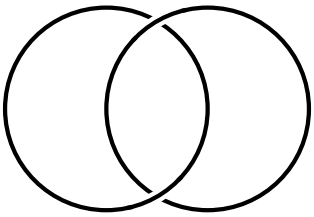
Solution: Note that whether crossings are over or under does not affect whether a knot diagram is checkerboard colorable. Thus, we can assume by Question 4.3 that the crossings are resolved in a manner so that the knot is equivalent to the unknot. In other words, this knot diagram can be achieved by performing some series of Reidemeister moves on the unknot. Since the zero-crossing unknot diagram is checkerboard colorable, and checkerboard colorability is invariant under Reidemeister moves, any knot diagram is checkerboard colorable.

Question 4.9. (3 pts) Consider a reduced knot diagram with a checkerboard coloring. Prove that the process of performing RCC moves on all shaded regions in the checkerboard coloring will result in the original reduced knot diagram.

Solution: In a reduced diagram, no crossing is reducible, so the four regions adjacent to each crossing are all different regions. Moreover, among the four regions around each crossing, exactly two must be shaded for checkerboard coloring. Thus, the process of doing RCC moves on all shaded regions will perform two crossing changes on each crossing. Since flipping a crossing twice results in the original crossing, no crossings are changed, so the knot diagram is unchanged.

For the rest of the section, we'll focus on proving that RCC is also an unknotting operation. In particular, given a knot diagram and any crossing c in the diagram, we can perform a series of RCCs so that c has its crossing changed, but no other crossings are changed. Since the crossing change is an unknotting operation, the RCC is also an unknotting operation.

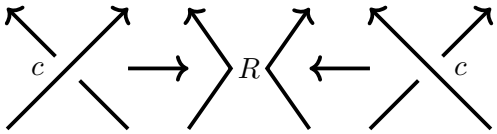
Definition 4.6. A **link** is a collection of closed loops. Each closed loop in the link is a **component** of the link. For example, the link diagram below has two components.



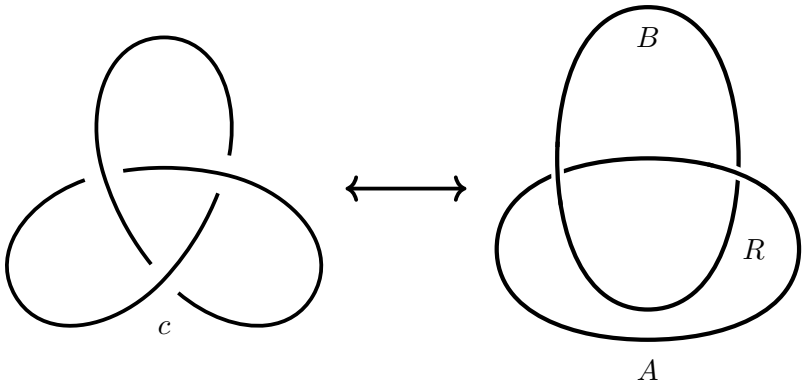
Recall from Chapter 2 that we can trace along a knot in a particular direction. We can use arrows to denote the direction we are moving along the knot in. This notion of direction is also useful for unknotting, as we see below.

Definition 4.7. A **splice** is the act of removing a crossing from an oriented knot diagram with respect to the orientation of the diagram. Examples of splices are shown below. Note that the orientation of the diagram should be preserved.

The diagram below shows a splice on crossing c. Region R is the region where crossing c was located before the splice. Note that the arrows denoting orientation are in their original positions after the splice.



A splice on crossing c in the trefoil is shown below. Region R is the region after the splice where crossing c used to be. Note that after the splice, the trefoil is split into a link diagram with components A and B.



Question 4.10. (4 pts) Prove that a splice on a knot diagram will always convert it to a link diagram with two components.

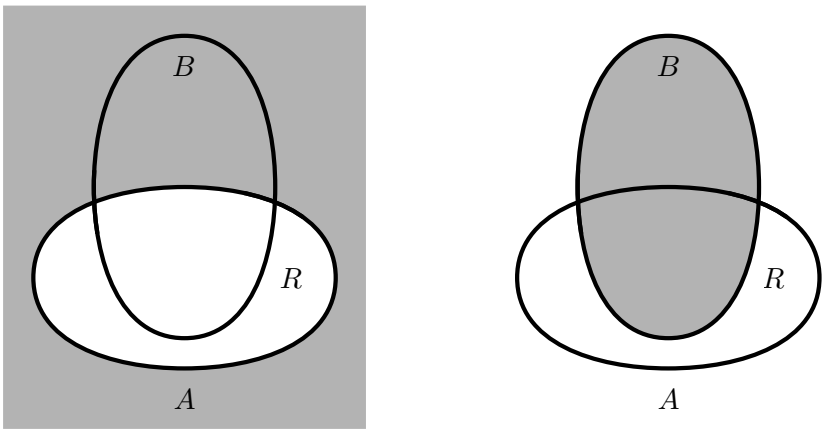
Solution:

Suppose we move along the knot along its orientation. For a given crossing c, consider the part of the knot traveled between the first time we encounter that crossing and the second time we encounter that crossing. Let A denote the part of the knot as described above, and let B denote its complement (i.e., the part of the knot that is not A). Both are nonempty arcs. Traversing the knot once, we pass through c exactly twice, so A starts where we leave c the first time and ends where we arrive at c the second time, while B starts where we leave c the second time and ends where we arrive the first time. A splice at c respects orientation, so it joins the end of A to the start of A, and the end of B to the start of B. Thus A and B each become a closed loop, and there is no way to get from A to B or from B to A, so they are separate components. (They may still cross each other at other crossings, but a crossing does not connect the two strands.)

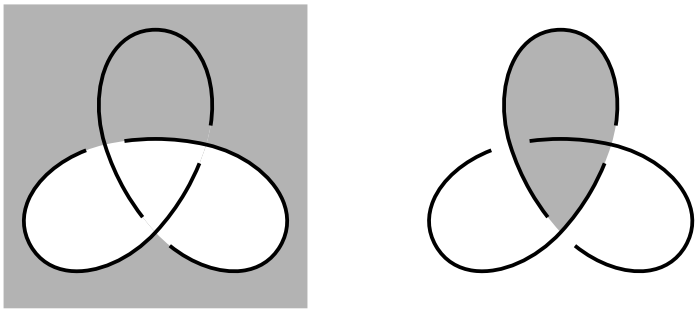
It is not obvious how to determine the regions on which we should perform RCCs. To start, let's consider the following three-step operation:

1. Splice a crossing c in the knot diagram. Let R denote the connected region from this splice.
2. On the resulting diagram, checkerboard color one of the two components while ignoring the other. Choose the checkerboard coloring that does not shade in the region R.

   The diagrams below show two ways to checkerboard color the spliced diagram of the trefoil. The diagram on the left shows when component A is colored. The diagram on the right shows when component B is colored. Note that region R is unshaded in both cases.



3. Undo the splice, remembering which regions were shaded in the checkerboard coloring. Then, perform an RCC move in all of the shaded regions in the unspliced knot diagram. For instance, the diagrams below depict the unspliced versions of the links from the examples above. Note that region crossing changes will be performed on the shaded regions.



Question 4.11. (8 pts) Prove that performing the three-step operation above on crossing c in a knot diagram is equivalent to a crossing change on c if all crossings other than c are not reducible.
Solution: Without loss of generality, suppose that link component A from the three-step operation is checkerboard colored, while component B is not.

For crossing c, the four regions around c are two regions that will form region R, a region from A, and a region from B. From the three-step operation, we are given that R is always unshaded. Moreover, the region exclusive to B adjacent to c is unshaded as well, since B is ignored in the coloring and so this region has the same shade as R. Finally, for component A, since R is unshaded, component A's other region adjacent to c must be shaded. Thus, exactly one region around c is shaded, so performing RCC moves on all shaded regions will always result in c getting a crossing change.

All other crossings are not reducible and can be classified into three groups:

- Both strands of the crossing are from component A.
- Both strands of the crossing are from component B.
- One strand is from component A, and the other strand is from component B.

In general, if an even number of distinct regions around a crossing are shaded, doing RCC moves on all shaded regions will result in the original crossing. On the other hand, if an odd number of distinct regions around a crossing are shaded, performing RCC moves on all shaded regions will result in the crossing being flipped.

For crossings where both strands are from component A, two opposite regions will be shaded since A is checkerboard colored. These must be different regions since those crossings are not reducible, so exactly two different regions around those crossings are shaded, and performing RCC moves on the shaded regions results in the original crossing.

For crossings where both strands are from component B, all four regions around those crossings will be either shaded or unshaded, since the strands of B have no effect on the checkerboard coloring of A. Thus, either 0 or 4 distinct regions around those crossings are shaded, since those crossings are not reducible. Since 0 and 4 are even, performing RCC moves on those regions will result in the original crossing.

For crossings where one strand is from A and the other is from B, exactly two adjacent regions will be shaded: the two regions on the shaded side of the strand from A. Since the crossing is not reducible, these are two distinct regions, so doing RCC moves on each region will result in the original crossing.

As shown above, the three-step operation can be used to successfully unknot any reduced knot diagram. Now, we consider knot diagrams that still have reducible crossings.

Question 4.12. (6 pts) Suppose a knot diagram D has exactly one reducible crossing, d. Prove that for any other crossing, c, a series of RCC moves exists that results in a crossing change on c and no other crossings.
Solution: Note that since d is the only reducible crossing, if we perform the three-step operation on d, the only crossing that is flipped is d. In particular, if $c = d$ we are done.

Now suppose $c \neq d$. In order to flip c, and no other crossings, we can perform the three-step operation on c. As shown in Question 4.11, c will be flipped, and none of the crossings other than d will be flipped, since none of those crossings are reducible. Afterwards, if d is flipped, do the three-step operation on d, which flips only d. Otherwise, we are done since only c was flipped.

For knot diagrams with more than one reducible crossing, we can prove that there exists a series of RCC moves equivalent to any crossing change through induction on the number of reducible crossings. Question 4.11 establishes the case of zero reducible crossings, and Question 4.12 handles one reducible crossing.

Suppose we are given that for some nonnegative integer k, for knot diagrams with up to k reducible crossings, for any crossing c, there exists a series of RCC moves that will cause a crossing change at c without changing any other crossings. We now want to prove the case for up to $k+1$ reducible crossings.

Consider an arbitrary knot diagram with $k+1$ reducible crossings. We want to perform a crossing change on c (which may or may not be reducible) through a series of RCC moves without changing any other crossings.

Question 4.13. (4 pts) Prove that if a knot diagram is spliced at a reducible crossing, the two resulting components do not have any intersections in the diagram.
Solution: Suppose that we use the region labeling conventions from Question 4.6, where the regions adjacent to a crossing are A, B, C, D, labeled so that the splice merges A and C into one region R, while B and D end up on opposite sides of the spliced crossing. The same region can only appear at two opposite corners, so the reducible pair is either $\{A, C\}$ or $\{B, D\}$. It cannot be $\{B, D\}$: a path from B to D avoiding the knot, closed up through the splice site, would be a closed curve crossing each of the two components exactly once, which is impossible. So A and C denote the same region. In this case, from one point in A, we can draw a path to a point in C without crossing the knot. Thus, after splicing, we can complete this path through R, creating a closed loop that does not meet the spliced diagram. The loop separates B from D, so the component next to B is enclosed by this path, while the component next to D is not. Since this closed path separates the two link components, they don't intersect at any point.

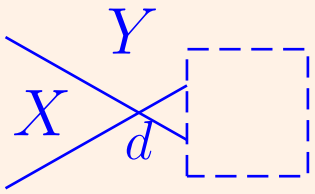
Question 4.14. (12 pts) Prove that for all integers $k \geq 1$, on any knot diagram with $k+1$ reducible crossings, there exists a reducible crossing $p \neq c$ such that splicing at p will result in a component with c and k reducible crossings, and another reduced component that does not include c or any reducible crossings. (Hint: First prove that there is a crossing p where the diagram can be spliced into one component with k reducible crossings and one component with zero reducible crossings, without considering where c is. Then, if c is not on the component with k reducible crossings, devise a way to find a different reducible crossing q where c is on the component with k reducible crossings.)

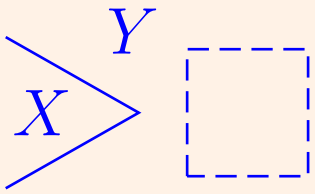
Solution: Start at some arbitrary reducible crossing $d \neq c$ and consider the result of splicing on it. As shown in Question 4.13, the two resulting link components do not intersect, so the other k reducible crossings are split between them. Without loss of generality, suppose that crossing c is on link component A, which has a reducible crossings, while link component B has $b = k - a$ reducible crossings. If $a = k$, then B is reduced and we are done. If $a < k$, then $b \geq 1$, so we choose a reducible crossing d' of B and splice the original diagram at d' instead. The new component containing c contains all of A, including its a reducible crossings, and also contains d, which is reducible, so it has at least $a+1$ reducible crossings. Since this strictly increases the number of reducible crossings on the component with c, we can repeat this with the new components, and since this number is at most k, we eventually reach a splice where the component with c has exactly k reducible crossings, meaning the other component is reduced.

Question 4.15. (16 pts) Prove that for any crossing c on any knot diagram with $k+1$ reducible crossings (where $k \geq 1$), there exists a series of RCC moves that changes the crossing on c and does not change any other crossings, assuming that this statement holds for all knot diagrams with k reducible crossings.

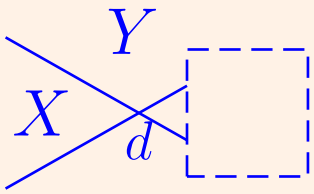
Solution: By Question 4.14, we can splice the knot (on some reducible crossing d) into a link component we'll call A, with c and k reducible crossings, and a reduced component which we'll call B. We can use the inductive hypothesis on the knot diagram A to get a set of regions that flips c and no other crossings on A. Thus, we only need to consider how the set of regions needs to be modified to ensure that d and the crossings on B remain unchanged. The scope of this proof consists of X, the region in A that bordered d before the splice, and Y, the region surrounding B, as well as all the interior regions of B, since the strands of B are not considered when finding regions to flip crossing c in A. X and Y are labeled below. Part of component A is on the left, and reduced component B is contained within the dashed box. Before splicing, crossing d joins component B with the rest of the knot.

Below, the left diagram is with crossing d included, and the right is after splicing d.



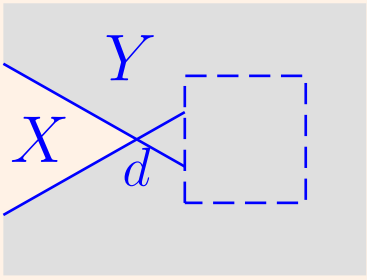


Since Y is either completely shaded or completely unshaded (including the dashed box), each crossing within the reduced component B is surrounded by 0 or 4 shaded regions, so they will not be flipped if RCC moves were performed on shaded regions. (In cases 3 and 4 we recolor B; then each crossing of B is surrounded by exactly 2 distinct shaded regions, since B is reduced, so it is still not flipped.) Thus, we focus on changing the region shading to ensure d is not flipped.

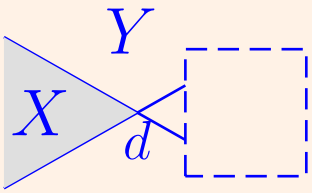


The four regions around d are X, Y (above and below d, which are the same region), and the region to the right of d bordering B, which we call the region opposite X.

1. If X and Y are unshaded, then d has 0 adjacent shaded regions, so d is not flipped and no changes need to be made.

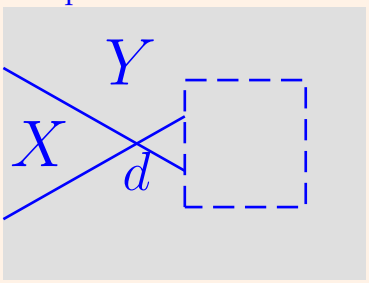


2. If X is unshaded, but Y is shaded, d appears to be surrounded by 3 shaded regions (above, below, and to the right of d), but two of them (above and below d) are the same region, so d is only adjacent to 2 distinct shaded regions. Thus, d will not be flipped by the current set of shaded regions, so no changes need to be made.



3. If X is shaded, and Y is unshaded, then d is only adjacent to one shaded region. Thus, we need to modify the shading, or else d will be flipped if we perform RCC moves.

   The solution to this is to shade the region opposite X and checkerboard color component B accordingly. The crossings in B will still not be flipped since they are adjacent to 2 distinct shaded regions. The crossings in A are unchanged since the region exterior to B does not have its shading changed. d is now adjacent to 2 distinct shaded regions, so it, too, is no longer flipped when RCC moves are performed on all shaded regions.



4. If X and Y are shaded, then d is surrounded by 3 distinct shaded regions. To alleviate this, we unshade the region opposite X, and accordingly checkerboard color the rest of component B. As mentioned in the previous case, crossings in B and A are unchanged by this operation. However, this reduces the number of distinct shaded regions around d to 2, so d is no longer flipped when RCC moves are performed on all shaded regions.

If you're interested in this topic, we suggest reading the research paper *Region crossing change is an unknotting operation* by Ayaka Shimizu, linked here.

Then by induction, on any knot diagram with an arbitrary number of reducible crossings, there exists a set of RCC moves to make a crossing change on any crossing c and no other crossings. So, we can unknot any knot diagram with RCC moves by first finding the set of crossing changes needed, and then performing RCC moves to get those crossing changes. In other words, we have found that RCCs are simple but surprisingly powerful operations for transforming knots.

Congratulations on finishing the BMT 2025 Power Round! We hope that you enjoyed exploring knot theory and are excited to learn more about it in the future. Good luck on your next tests, and have fun!