0 Preliminaries

The theme of this power round is origami! Before beginning, we introduce some rules and guidelines regarding submissions to certain problems.

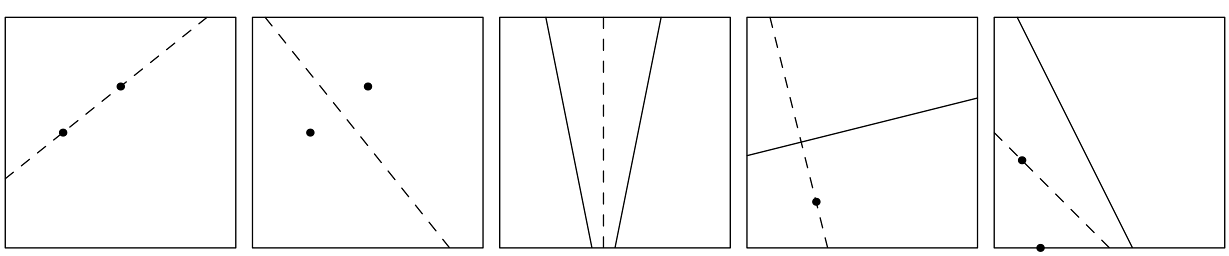
Recall that, as stated in the instructions, problems that use the words "construct" or "fold" take origami submissions. To submit your answers for any origami constructions, you may do either of the following:

- Submit a **folded** piece of origami, with all intentional creases also **clearly marked and labeled with pen or pencil**. See the crease pattern at the bottom of page 4 to see what the origami should look like when unfolded. The problem number must be clearly written somewhere on the piece. In addition, number every fold in the order of folding.
- List a sequence of folds with diagrams as needed on your submission paper. Each fold should be labeled (A) through (E) as described below, and the ordering of the folds should be clearly labeled. For example: "Step 1: Fold the top-left corner to the bottom-right corner using fold (B)." If a fold creates multiple crease lines, label each crease line with the same number in the sequence.

The rules for folding submittable origami are as follows:

- **A fold on the paper will create one or more *crease lines*, which are lines created by paper being folded onto itself. These lines, as well as their intersections with other lines, may be used as reference points in later steps.**
- **You can apply folds to an already-folded paper, but you may only make one crease at a time.**
- **Unfolding along existing crease lines does not count as a fold.**
- The following rules describe all "allowable" folds for origami submissions in Chapters 1 and 2:
  - **(A)** If you have two known points, you may create the fold passing through both points.
  - **(B)** If you have two known points, you may create the crease that folds one point onto the other.
  - **(C)** If you have two known lines, you may fold one line onto the other.
  - **(D)** If you have a point and a line, you may create the fold perpendicular to the line, passing through the point.
  - **(E)** If you have two points and a line, you may fold one of the points to the line via a crease that passes through the other point.
- In particular, you may not:
  - Fold a point arbitrarily onto a line
  - Fold a line arbitrarily onto a point
  - Make two folds simultaneously (e.g. folding a paper into thirds with one "fold")

Example Folds:



(A) (B) (C) (D) (E)

1 Constructions (40 pts)

Before beginning the round, please make sure you have carefully read the Preliminaries on the back of the cover sheet.

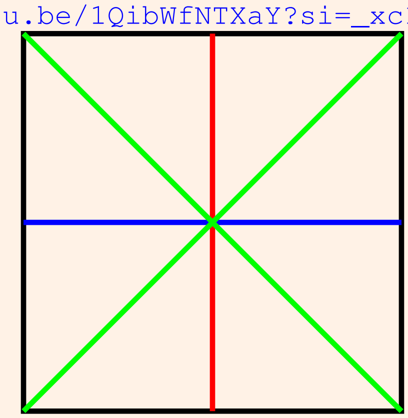
Welcome to the power round! Let's take some time to explore something you might have become familiar with years ago: origami. Origami is a source of several interesting problems in mathematics, with deep connections to topics like graph theory, solving polynomials, and field theory.

In this section, let's think about how origami can be used to build upon the classic constructions of a straightedge and compass.

Question 1.1. (3 pts) In the fewest number of folds possible, fold a paper so that when it is unfolded, the crease lines divide the paper into eight congruent triangles. Additionally, explain why this is the fewest number of folds possible. (Hint: It's not 4.)

Solution:

https://youtu.be/1QibWfNTXaY?si=_xc1uKIbNyUDnrj-



Each fold can divide a face of the paper into at most 2 parts. So, the minimum possible number of folds is $\log_2(8) = 3$, and this is achievable by folding vertically in half, then horizontally in half, then folding the resulting square diagonally in half.

Instructions:

- Fold one side to the other, creating a crease line dividing the paper in half using crease **C**. Keep the paper folded.
- Fold one side to another again with **C**, creating a crease line dividing the rectangular paper in half to create two squares.
- Divide the resulting square into two triangles by folding opposite corners to each other (**B**).

These folds may be done in any order, as long as the origami is not unfolded before the end.

Constructions can be very difficult, so it is often useful to first consider the geometry of the problem. Ask yourself what shapes or segments would be useful to have in order to make the construction. Let's break down our first construction into a few different parts. In the next few questions, we will be constructing a square with $3/4$ the area of our original square.

Question 1.2. (2 pts) Given a square with side length l and area A, find the side length x of a square with area $\frac{3}{4}A$ in terms of l.
Solution: We solve $l'^2 = \frac{3}{4}A$.

$$l^2 = \frac{3}{4}A \implies l = \sqrt{\frac{3}{4}A} = \sqrt{A}\sqrt{\frac{3}{4}} = l\frac{\sqrt{3}}{2}.$$

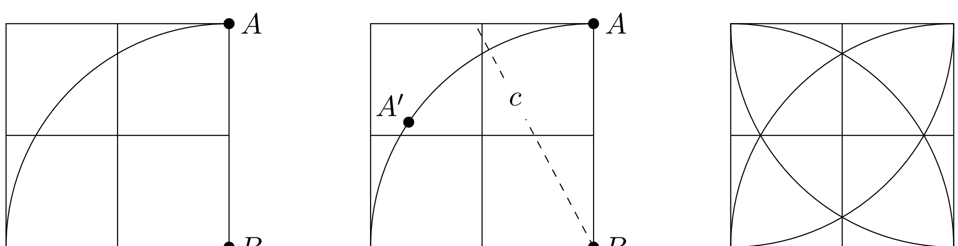
The next thing we want to think about is how we could construct the side length we want. While there are many different shapes that could be folded, we focus on the simplest ones.

Question 1.3. (2 pts) Show there exists a right triangle with side lengths $x, l, l/2$.
Solution: A right triangle with hypotenuse l and leg $l/2$ will have its other leg as length $l\frac{\sqrt{3}}{2}$ by the Pythagorean Theorem, or by noticing that these are the side lengths of a 30-60-90 triangle.

Our goal now is to make folds that correctly measure the above lengths at the correct angle. Let's consider the geometry of the square and make some conclusions about where the triangle must be located. Importantly, where must a vertex of the side with length l be, and how can you measure length $l/2$?

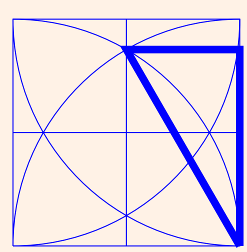
The only way to create a length l ending at a vertex B of the square is to rotate a side length $\overline{AB}$ about the vertex B. To do this, we can reflect the vertex A across a line c through B, creating A'. Conveniently, this is the same thing as creasing along c! The set of all points length l from a vertex is just the circle with center B, and radius l. Considering only the parts of these circles intersecting the paper, we note that all points on the following arcs are (theoretically) constructible, but we don't have enough information to accurately fold all of them. Now, we want to create $l/2$. The only length we have now is l, so we need to divide this in half. We can do this by folding horizontally and vertically in half. These lines will be distance $l/2$ from the parallel sides of the square.

As a corollary, if A rotates by an angle θ, the crease line will form an angle $\theta/2$ with the original line $\overline{AB}$.



(a) (b) (c)

Question 1.4. (2 pts) Draw a possible orientation and location of the right triangle found in Question 1.3 with vertices on the lines and curves given in diagram (c).
Solution: Answers may vary, must be 30-60-90 triangle with 30° angle at a vertex of the square and right angle along a side of the square.

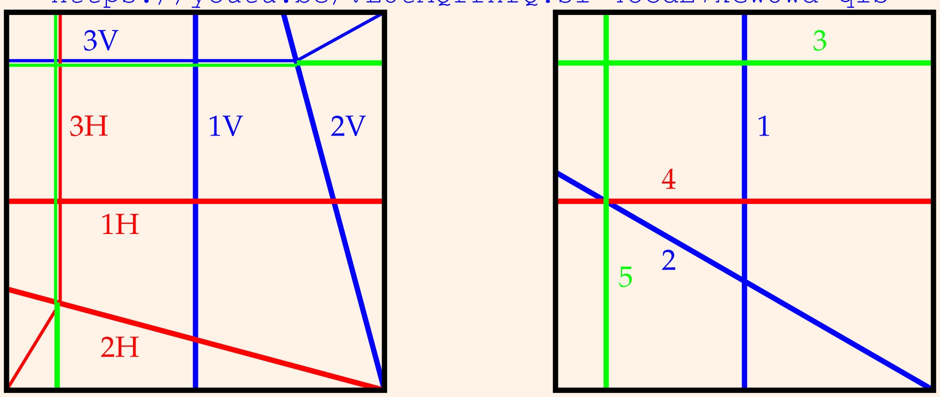


We now have a possible diagram for our construction of a segment with length x. Now it's time to determine whether this diagram is foldable.

Question 1.5. (5 pts) Given a square piece of paper, construct a square with exactly $3/4$ the area of the original in at most eight folds.

Solution:

https://youtu.be/vL8tAQffnfQ?si=4oCdE7kGwUwd-qlS



As in the video, there are at least two solutions that may be combined. The crease types used are noted below, but keep in mind there are other orderings/solutions that may not match exactly.

First way:
- Fold vertical midline + unfold (**C**)
- Fold the top right corner to the midline, with the crease passing through the bottom right corner (**E**)
- Fold the top of the paper down, with the crease passing through the image of the top right corner (**D**)
- (Unfold everything (not a "real" step))
- Repeat 1-4 using the horizontal midline and bottom left corner instead
- Fold the creases formed in step 3 down to create a square (**D**)

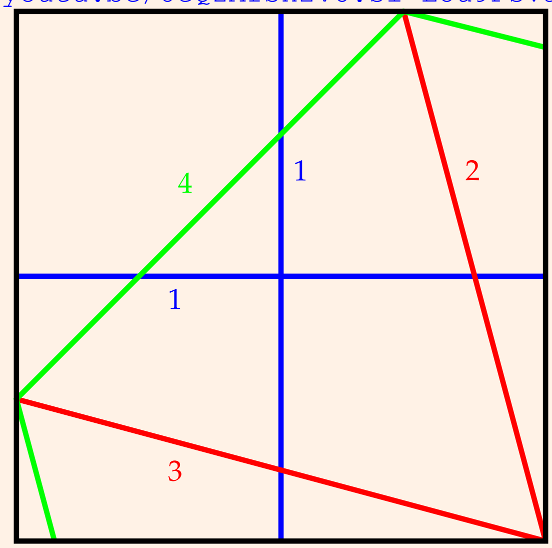
Second way:
- Fold vertical midline + unfold (**C**)
- Fold the bottom left corner to the midline, with the crease passing through the bottom right corner (**E**)
- Fold the top of the paper down, with the crease passing through the image of the bottom left corner (**D**).
- Unfold only step 2
- Fold horizontal midline (**C**) and unfold.
- Fold the left side of the paper inwards, with the crease passing through the intersection of the horizontal midline and crease made in step 2 (**D**).

Origami can be used to create other interesting polygons as well.

Question 1.6. (5 pts) Given a square piece of paper, draw an equilateral triangle with the greatest possible area and prove that its area is indeed maximal. Additionally, fold this triangle.

Solution:

https://youtu.be/8eQzAfSn2v8?si=LOu9PsVat0gdD3Y7



Intuitively, we can deduce that all vertices must be on the edges of the square (since otherwise we could move the triangle to have it touching no sides of the square, which means it could then get larger). Further, we can see that the triangle must actually share a vertex with the square for similar reasons (you would be able to slide one vertex away the side of the square). From there, the triangle must be symmetric with respect to the diagonal of the square through that vertex (in order for the distances from the shared vertex to be equal). The symmetry then gives us a triangle with sides at 15 degree angles from the sides of the square at the shared vertex. We then place the two remaining vertices where these rays from the corner intersect the square again, and we have constructed a triangle with maximal area.

1. Fold horizontal and vertical midlines, unfolding after each (**C**)
2. Fold top right corner to vertical midline, crease passing through bottom right (**E**)
3. Fold bottom left corner to horizontal midline, crease passing through bottom right (**E**)
4. Crease through the new top right and bottom left corners, creating the equilateral triangle (**A**)

Notice that in these cases, the main challenge was constructing a segment of a certain length. This is a way of thinking about origami that may be new, so let's try to make some other segments as well. Importantly, we want to try to make segments of arbitrary rational length a/b. These measurements are quite useful to people actually making origami, so it's good to have a way to construct them.

The main difficulty is constructing a segment of length $1/n$, where n is odd. For $a > 1$, we can just take a lengths of $1/n$, and dividing a length in half is easy (we can just fold in half), so a length $\frac{1}{2k}$ for some k is trivial after $1/k$ is created.

We begin our construction of segments with an example that may appear trivial: dividing a paper into equal thirds. However, the "classic" method of lining up both folds at once and pressing the paper flat violates one of our rules: that only one fold may be made at a time.

Question 1.7. (6 pts) Fold a piece of paper into thirds in five folds, making exactly one fold at a time. *(Hint: create two similar triangles with side length ratio 1:2)*

Solution: We present a solution in five folds; there is another approach that only requires four.

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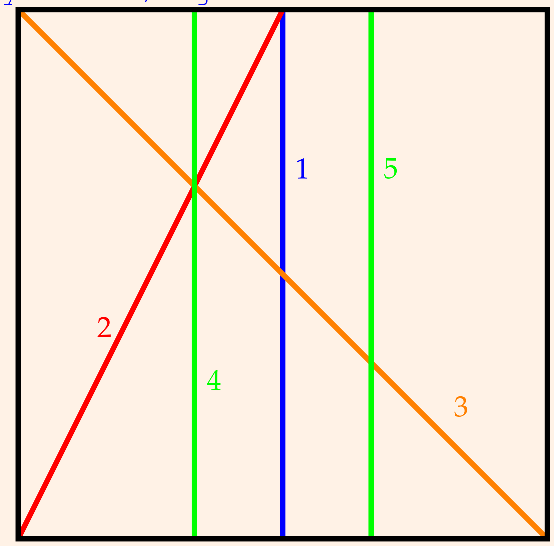


- Fold vertical midline (**C**) and unfold.
- Fold crease between bottom left corner and top of midline (0.5,1) (**A**) and unfold.
- Fold diagonal from top left to bottom right (**A** or **B**) and unfold.
- Fold horizontal (or vertical) line through the intersection of folds 2 and 3 (**D**)
- Fold the other third inwards (doable by **all folds**)

Before diving into the main problem of a/b, let's work on a more specific case. Some side lengths are easy to create (1/3, 1/5, 1/7, for instance), so let's try to use these to create some more segments. Specifically, we want to create a segment of length $\frac{1}{2k-1}$ given a segment of length $\frac{1}{2k}$.

Note that $[0,1] \times [0,1] \subset \mathbb{R}^2$ is the coordinate plane with x- and y-values between 0 and 1 inclusive.

Question 1.8. (3 pts) Given a square sheet of origami paper defined as $[0,1] \times [0,1]$, let $A = \left(0, \frac{1}{2k}\right)$ and $B = \left(\frac{1}{2k}, 1\right)$. Now, fold A onto B, with the crease line L intersecting the y-axis at point $P = (0, P)$. Prove that the distance from P to $(0,1)$ is $\frac{k-1}{2k-1}$.

Solution: Note that the crease is the perpendicular bisector of the line $\overline{AB}$. Notice $\overline{AB}$ has slope $(1-1/2k)/(1/2k) = (2k-1/2k)/(1/2k) = 2k-1$, and the midpoint of this line is $(((1/2k)/2), ((1+1/2k)/2)) = (1/4k, (2k+1)/4k)$. In point-slope form, the perpendicular bisector has equation

$$y - \frac{2k+1}{4k} = \frac{-1}{2k-1}\left(x - \frac{1}{4k}\right)$$

When $x = 0$,

$$y = \frac{1}{(2k-1)(4k)} + \frac{2k+1}{4k} = \frac{1+(2k-1)(2k+1)}{(2k-1)(4k)} = \frac{4k^2}{(2k-1)(4k)} = \frac{k}{2k-1}$$

The distance from $\left(0, \frac{k}{2k-1}\right)$ is $1 - \frac{k}{2k-1} = \frac{k-1}{2k-1}$.

That was certainly not clear to begin with, but with this fold we can now generalize to a length $\frac{a}{2k-1}$.

Question 1.9. (4 pts) Given an algorithm for dividing a segment into k segments of equal length, give an algorithm for constructing a length $\frac{a}{2k-1}$ for any integer a between 1 and $2k-1$.
Solution: Construct the points $(0, 1/2k)$ and $(1/2k, 1)$ by creating $1/k$ and then folding in half on the respective sides. Then, make the fold detailed in Question 1.8 to create $P = (0, k/(2k-1))$. We use our algorithm again to divide the segment between $(0,0)$ and P into k segments of length $1/(2k-1)$ to create $P_1 = (0, 1/(2k-1))$. If $P_0 = (0,0)$, we can create P_i from P_{i-1} and P_{i-2} by folding P_{i-2} over a crease perpendicular to the left edge of the paper through P_{i-1}, which allows us to create the point $P_a = (0, a/(2k-1))$. Then, we have constructed the desired length from $(0,0)$ to P_a.

This algorithm, while weaker than a general rational length generator, will give us insight into how we can create a more general algorithm.

Question 1.10. (8 pts) Given a square sheet of origami paper defined as $[0,1] \times [0,1] \subset \mathbb{R}^2$, find, with proof, an algorithm to create the point $(0, a/b)$.
Hint: A length we can always create is $b/2^{k+1}$, where 2^k is the largest power of 2 smaller than b.
Solution: We proceed by induction.
Base case: The length $1 = 1/2^0$ is constructed.
Inductive Hypothesis: For $1 \le n \le 2^m$, we can construct $1/n$.
Inductive Step (induction on m): We want to show that $1/n$ for $2^m < n \le 2^{m+1}$ is constructible. We proceed by cases.
Case 1: If n is even, then $n = 2k$ for some $k \le 2^m$. Then by our hypothesis $1/k$ is constructible, so $1/2k = 1/n$ is constructible as well by bisecting the segment.
Case 2: If n is odd, then $n = 2k-1$ for some $k \le 2^m$. Since $1/k$ is constructible by our hypothesis, we use the algorithm in Question 1.9 to construct $1/(2k-1)$ as requested.

2 Flat Folding (53 pts)

Now that we've spent some time considering constructible segments and shapes using origami, let's dive into the theory behind origami, and figure out when different origami patterns have certain properties. The main focus of this section will be considering when origami can fold flat. Determining whether a random assortment of creases folds flat is an extremely difficult task (NP-hard, in fact), but we can take a look at some specific cases.

Before we can start, let's go over some quick graph theory and origami definitions.

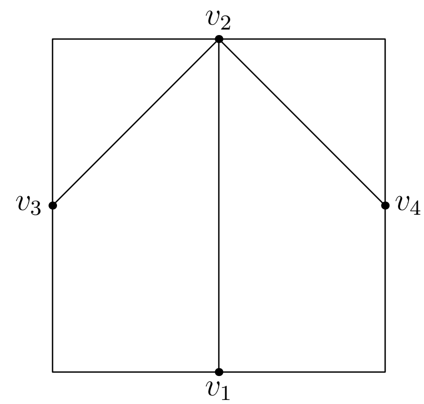
Definition 2.1. A **graph** $G = (V, E)$ is a set of vertices, V, and a set of edges, E. Each edge is itself an (unordered) set of two distinct vertices. Vertices that share an edge between them are called **adjacent**. The number of vertices in a graph is denoted $|V|$ and the number of edges is denoted $|E|$. The **degree** of a vertex is the number of edges connected to the vertex.

Definition 2.2. Given a piece of paper $R \subset \mathbb{R}^2$, a **crease pattern on** R is a graph $G = \{V, E\}$ with vertices V and creases E. Vertices on the boundary of R are called **boundary vertices** and vertices in the interior of R are called **interior vertices**. The **faces** of the crease pattern G are polygons separated by the crease lines E. Note that only final creases must be in the crease pattern– if an intermediate crease is used but ends up not folded, it is not part of the crease pattern.

Definition 2.3. A **mountain fold** is a crease on a piece of paper which, when viewed from directly above the crease, makes the crease lie above the faces on either side. Intuitively, it is a fold that creates a "mountain" shape.
A **valley fold** is a crease on a piece of paper which, when viewed from directly above the crease, makes the crease lie below the faces on either side.
A valley fold is the opposite of a mountain fold, in that a mountain fold will look like a valley fold when viewed from below and vice versa.

Note that the paper is defined as a subset of $\mathbb{R}^2$, but it is not necessary to define each point as a coordinate pair when simply drawing a crease pattern. Below is the crease pattern after the first step of a simple paper airplane on paper $[0,1] \times [0,1]$.

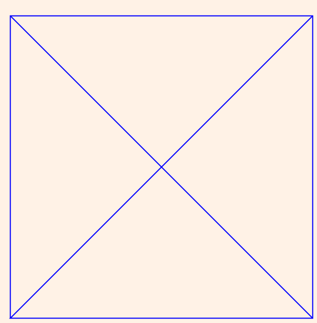
$$V = \left\{ \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{2}\right) \right\}$$
$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}\}$$



Question 2.1. (2 pts) Given a square paper with the bottom left corner at $(0,0)$ and the top right corner at $(1,1)$, give a crease pattern G with exactly one interior vertex and four edges, and draw the crease pattern.
Solution: Answers may vary. We fold in half along both diagonals.

$$V = \{(0,0), (1,0), (0,1), (1,1), (\mathbf{0.5}, \mathbf{0.5})\}$$

$$E = \{\{v_1, v_5\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$



Now, let's introduce the main topic of this section: folding origami flat. Intuitively, some crease patterns will be able to be folded flat and some will not be. How can we determine what is possible and what isn't?

Definition 2.4. A crease pattern $G = (V, E)$ **folds flat** if, after folding along all crease lines and nowhere else (creating no new creases), the resulting origami would be a two-dimensional polygon if the paper had zero thickness.

Now that we know how to write crease patterns and what it means for something to fold flat, let's think about how we could formalize the idea of folding along these lines to create origami. The two most basic ways to fold paper are *mountain folds* and *valley folds*, and we will define a way to assign these folds to our crease patterns now.

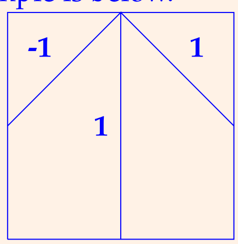
Definition 2.5. A **mountain-valley assignment** (or **MV-assignment**) for a crease pattern $G = (V, E)$ is a function $\mu : E \to \{-1, 1\}$ that assigns each crease line c in E a folding angle of $\mu(c)\pi$ from the horizontal. An MV-assignment is *valid* if the crease pattern can be folded flat.

Intuitively, a crease is a mountain fold if it is assigned -1 under μ and a valley fold if it is assigned 1.

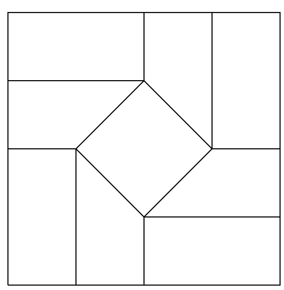
For our purposes, it is sufficient to simply label each crease in a crease pattern with 1 or -1.

Question 2.2. (2 pts) Does a valid MV-assignment exist for the paper airplane crease pattern given after Definition 2.1? If so, give a valid MV-assignment. If not, explain why.

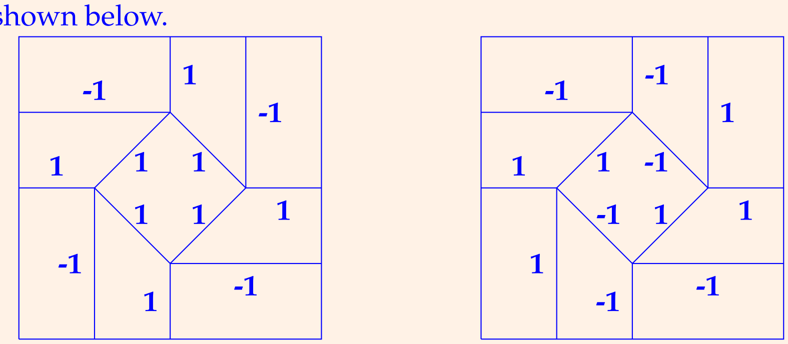
Solution: Any MV-assignment is valid, as these folds do not interact with each other. We can label each crease with either 1 or -1. An example is below.



Question 2.3. (4 pts) Does a valid MV-assignment exist for the **square twist** crease pattern shown below? If so, give a valid MV-assignment. If not, explain why. *(Hint: Actually try to fold the shape!)*



Solution: There are two valid MV-assignments up to rotation and inversion of mountains and valleys. Both are shown below.



Question 2.4. (3 pts) Explain why the faces of a flat-foldable origami crease pattern must be 2-colorable: that is, each face can be assigned one of two colors in a way such that no two adjacent faces will have the same color. A rigorous proof is not necessary.
The reverse (that a 2-colorable origami crease pattern is flat-foldable) is not necessarily true.
Solution: When we fold a shape flat, each face of the paper is either oriented up or down. If two faces share an edge (a crease), then they have to be pointed in opposite directions, since the fold "flips" the paper. So, we fold the crease pattern flat and color the upward-oriented faces one color and the downward-oriented faces the other.

Before we continue, let's note some potentially useful proof techniques while working with origami. Proving theorems about origami will be very different than other subjects you may be familiar with. It may also valuable to consider "tracing" the cross-section of folded origami. Drawing a circle of radius 1 around a flat vertex fold (scaling if needed) and analyzing the image of this circle after folding the vertex flat may also give insights into the shapes. What directions does the curve go? What distance must it travel?

Now, let's turn our attention inwards and consider interior vertices. Flat-foldability is a very interesting property, so it stands to reason that we can make some observations about the interior vertices of a shape that folds flat. Let's narrow our scope a bit and think about single vertices.

Definition 2.6. A **single-vertex fold** is a crease pattern with exactly one interior vertex. A **flat vertex fold** is a single vertex fold that folds flat.

Notice that any crease pattern could be made up of many single-vertex folds.

Question 2.5. (3 pts) Prove that the degree (number of edges) of the interior vertex in a flat vertex fold must be even. (You may not use any results or theorems stated later in this section.)
Solution: Since the faces of the flat vertex fold must be 2-colorable and all faces touch the same vertex, there must be an even number of faces. Otherwise, two adjacent faces would be forced to have the same color, a contradiction.

We now know a lot about valid MV-assignments and have some criteria for flat-folding shapes. Let's use this to think about how mountain and valley folds must interact to fold flat. Here's a few tools that may prove useful.

Definition 2.7. Given a flat vertex fold $G = (V, E)$, let $E = \{l_0, l_1, ..., l_{2n-1}\}$ be the creases meeting at the interior vertex in clockwise order and let α_i be the angle between the creases l_i and l_{i+1} (α_0 is in between l_0 and l_{2n-1}). The **angle sequence** (α_i) of G is the sequence $(\alpha_0, ..., \alpha_{2n-1})$.

Lemma 2.8. Let $(\alpha_0, \alpha_1, ..., \alpha_{2n-1})$ be a sequence of $2n$ positive real numbers satisfying $\alpha_0 - \alpha_1 + \alpha_2 - \cdots - \alpha_{2n-1} = 0$. Then there exists an integer $0 \leq k \leq 2n - i$ such that $\alpha_k - \alpha_{k+1} + \alpha_{k+2} - \cdots \pm \alpha_i \geq 0$ for all $k < i \leq 2n - 1$ and $\alpha_k + \alpha_{k-1} - \alpha_{k-2} + \cdots \pm \alpha_i \geq 0$ for all $0 \leq i < k$.

Theorem 2.9. Kawasaki's Theorem: A single vertex crease pattern with angle sequence (α_i) folds flat if and only if

$$\sum_{i=0}^{2n-1} (-1)^i \alpha_i = 0.$$

That is, if and only if the alternating sum of angles is zero.

Lemma 2.10. (Big-little-Big Lemma) Let G be a flat vertex fold with angle sequence (α_i) and a valid MV-assignment μ. If, for some i, we have $\alpha_{i-1} > \alpha_i < \alpha_{i+1}$, then $\mu(l_i) \neq \mu(l_{i+1})$.

Theorem 2.11. Maekawa's Theorem: The difference between the number of mountain and valley folds in a **flat vertex fold** is 2.

Question 2.6. (12 pts) Prove Kawasaki's Theorem.
Solution: ($\implies$) Let G be a flat vertex fold, and consider the circle around the interior vertex of radius 1. If needed, scale G so this circle exists. Notice that the angles α_i are equal to their corresponding arc lengths. Now, let γ be the oriented curve on the boundary of our circle. After flat-folding, γ travels α_0 in the positive direction, then α_1 in the negative direction, and so on until it traverses arc α_{2n-1} in the negative direction to return to its starting point. We conclude that if G folds flat, $\sum_{i=0}^{2n-1}(-1)^i\alpha_i = 0$.

($\impliedby$) By Lemma 2.8, there is a k such that $\alpha_k - \alpha_{k+1} + \alpha_{k+2} - \cdots \pm \alpha_i \geq 0$ for all $k < i \leq 2n-1$ and $\alpha_k + \alpha_{k-1} - \alpha_{k+2} + \cdots \pm \alpha_i \geq 0$ for all $0 \leq i < k$. I provide the MV-assignment for crease l_i as follows:

$$\left\{\begin{array}{lll} -1 & k-i \equiv 0 & \pmod 2 \\ 1 & k-i \equiv 1 & \pmod 2 \\ 1 & k=i & \end{array}\right\}$$

This MV-assignment is valid. By Question 2.7, every partial alternating sum of the angles away from k are greater than 0. So, the arc swept out by each consecutive fold will remain on the same side of the point where l_k is located, and hence all faces will be "contained" within the valley crease at l_k. So, the "accordion" MV-assignment will be applied to all crease lines in a way that the faces will never collide with crease l_k, and hence it is a valid flat-fold.

Question 2.7. (3 pts) Prove that the sum of every other angle in a flat vertex fold is $180°$.
Solution: We consider the sum and difference of the even and odd-indexed terms.

$$\sum_{i=0}^{n-1}\alpha_{2i} + \sum_{i=0}^{n-1}(-1)^{2i+1}\alpha_{2i+1} = \sum_{i=0}^{n-1}\alpha_{2i} - \sum_{i=0}^{n-1}\alpha_{2i+1} = \sum_{i=0}^{2n-1}(-1)^i\alpha_i = 0.$$

Since the sum of all angles must be a full circle,

$$\sum_{i=0}^{n-1}\alpha_{2i} + \sum_{i=0}^{n-1}\alpha_{2i+1} = \sum_{i=0}^{2n-1}\alpha_i = 360°$$

We conclude

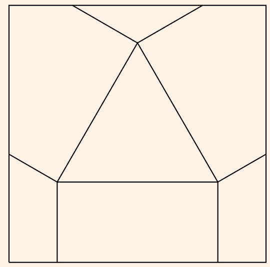
$$2\sum_{i=0}^{n-1}\alpha_{2i} = 360° \implies \sum_{i=0}^{n-1}\alpha_{2i} = 180° \text{ and } -2\sum_{i=0}^{n-1}\alpha_{2i+1} = -360° \implies \sum_{i=0}^{n-1}\alpha_{2i+1} = 180°$$

and we are done.

Question 2.8. (8 pts) Prove Maekawa's Theorem.
Solution: Consider the rotation of the paper's edge under a flat vertex fold. Let G be a flat vertex fold, and consider the circle around the interior vertex of radius 1. If needed, scale G so this circle exists. Now, we follow the oriented curve γ along the circle's edge. After folding, γ will rotate π radians when it reaches a mountain fold, and $-\pi$ radians when it reaches a valley fold. But, since γ is a closed curve, it must rotate a total of 2π radians. We conclude that $\pi M - \pi V = 2\pi \implies M - V = 2$, where M is the number of mountain folds and V is the number of valley folds. If γ was oriented negatively, the difference would be -2.

Recall our main goal: how can we determine what folds flat and what doesn't?

Question 2.9. (4 pts) Is the following crease pattern flat-foldable? If so, provide a valid MV-assignment. If not, explain why. This diagram is to scale. (You may not use any later results in the section.)



Solution: No, this is not flat-foldable. Let the creases of the equilateral triangle be l_1, l_2, l_3. By the Big-Little-Big Lemma, l_1 and l_2 must have different sign, l_2 and l_3 must have different sign, and l_1 and l_3 must have different sign. This is clearly impossible, as the third crease cannot be different from both of the other two.

We will now attempt to answer the question of flat-foldability for some specific crease patterns.

Definition 2.12. A crease pattern G is a **phantom fold** if all interior vertices satisfy the criterion described in Theorem 2.9 (Kawasaki's Theorem).

That is, a crease pattern is a phantom fold if each individual vertex could fold flat. It may or may not be globally flat-foldable, but this is definitely a step in the right direction when determining flat-foldability.

Let's now return to graphs.

Definition 2.13. Given a phantom fold $G = (V, E)$, the **origami line graph** $G_L = (V_L, E_L)$ is created as follows:
Let our initial set of vertices V_L be the midpoints of the creases $\{c_1, ..., c_n\}$ in G. Then,

- For each pair of creases $c_i, c_j \in E$, if they are forced to have different MV parity, let $\{c_i, c_j\} \in E_L$. (that is, connect the vertices associated with the two creases to each other).
- For each pair of creases $c_i, c_j \in E$, if they are forced to have the same MV parity and are *not* already the ends of a path of even length from performing the first step, add a new vertex $v_{i,j}$ to V_L and let $\{c_i, v_{i,j}\}, \{v_{i,j}, c_j\} \in E_L$.

Question 2.10. (4 pts) Verify that the crease pattern in Question 2.9 is a phantom fold and draw its origami line graph.
Solution: To show it is a phantom fold, consider each interior vertex. Notice that for each interior vertex, there is a pair of right angles opposite each other, so the alternating sum of angles must be zero (if the sum of the non-right angles is x, we have $2 \cdot 90 + x = 360 \implies x = 180$, so $90 - x + 90 = 0$) and therefore the interior vertices satisfy Kawasaki's Theorem and the crease pattern is a phantom fold. The origami line graph is a triangle (cycle with three vertices), which follows by Big-Little-Big lemma.

Finally, we are ready to make two important discoveries about flat-foldability!

Question 2.11. (3 pts) Prove that if the origami line graph G_L of a phantom fold G is not 2-vertex colorable, then G is not flat-foldable. A graph is 2-vertex colorable if each vertex can be assigned one of two colors in a way such that no two adjacent vertices will have the same color.

Solution: If G_L is not 2-colorable, there is a cycle in G_L with an odd number of vertices $2k+1$ (and hence edges). Since all edges imply that the two creases forming the edge have opposite MV parity, we may follow the cycle, alternating mountain and valley creases in for the creases in the cycle contained in G_L. If there are no additional vertices $v_{i,j}$ added in step (ii), μ must assign vertex i to $(-1)^i$. But, since vertex i is the same as vertex $i+2k+1$ (as it is contained in a cycle), it must also be assigned $(-1)^{i+2k+1} = (-1)^{i+1}$, the opposite parity. This is a contradiction, as the crease cannot be folded both ways.

We show that the vertices $v_{i,j}$ may be removed from the cycle when determining flat foldability. If a vertex $v_{i,j}$ exists in the cycle, two adjacent creases c_i, c_j have the same MV-assignment. We want to find our contradiction by considering the alternating MV-assignments, so we will ignore exactly one of these creases. Without loss of generality, we remove c_j, connecting c_j to c_{j+1} (which has opposite parity — if not, repeat this process on c_{j+1}). After removing all $v_{i,j}$ and relevant c_j, we have a cycle with alternating MV-assignments and an odd number of edges, so the above proof holds.

For some crease patterns, the origami line graph G_L completely determines MV-assignments. Because of this, we are encouraged to use these line graphs to make claims about the number of valid MV-assignments.

Question 2.12. (5 pts) Let C be a flat-foldable crease pattern with valid MV-assignments completely determined by C_L, and let n be the number of connected components of C_L. Find (with proof) the number of valid MV-assignments of C.

Solution: If two components of C_L are not connected, the MV-assignment of one is independent of the other's. Therefore, the number of valid MV-assignments is the product of the number of valid MV-assignments of each component. Now, notice that every crease in a connected component is dependent on the other creases in the component (either forced to be the same or different than the vertices it is connected to). So, choosing an MV-assignment for one crease will determine the MV-assignment for all of the connected component. There are 2 ways to do this ($+1$ or -1), so the number of valid MV-assignments is 2^n.

Recall that each valid MV-assignment is a unique way of folding a crease pattern. With this idea, we're now capable of figuring out how we can fold (some) crease patterns just by taking a good look at them!

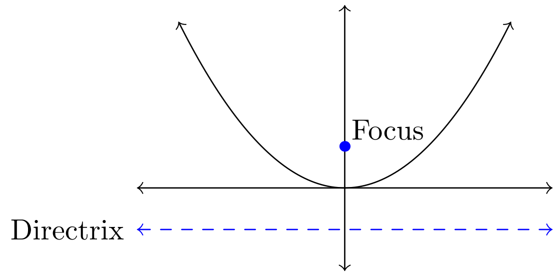
3 Solving Equations Using Origami (44 pts)

As it turns out, folding papers in certain ways can be used to solve polynomial equations.

Before getting into that, we'll first need to properly define a parabola. You may have seen that a parabola is a specific curve on a graph given by an equation like $y = x^2$.

In order to leverage origami theory to solve difficult equations, we'll need to use the rigorous geometric definition of a parabola, using the focus and the directrix.

Definition 3.1. A **parabola** is the set of all points in a plane that are equidistant from a fixed point, called the **focus**, and a fixed line, called the **directrix**.



In the case of the graph $y = x^2$, we see that our focus is $\left(0, \frac{1}{4}\right)$, and our directrix is $y = -\frac{1}{4}$. It turns out there's a pretty slick way to finding the focus and directrix for any parabola.

Theorem 3.2. For a quadratic that can be written in the form

$$(x - h)^2 = 4p(y - k)$$

for real numbers h, k and p, the focus of the corresponding parabola is $(h, k + p)$, and the directrix is $y = k - p$.

Question 3.1. (2 pts) Compute the focus and the directrix of each of the following two parabolas:

$$y = x^2 + 5, \tag{1}$$

$$16y - 3x^2 = 32. \tag{2}$$

Solution: Through a combination of equation manipulation as well as inspection, the focus and directrix for the first equation is $(0, 21/4)$ and $y = 19/4$. For the second equation, the focus is $(0, 10/3)$ and the directrix is $y = 2/3$.

Now that we've defined these terms, we can start diving deep into how origami theory fits in all of this:

Theorem 3.3. Folding a point P to a line L and then unfolding will create a crease line tangent to the parabola with focus P and directrix L.

By applying Theorem 3.3 again and again an infinite number of times, choosing different points on the left and right edges of the paper such that we make a crease at this point which will fold P onto L will eventually trace out a parabola on our paper with focus P and directrix L.

The proof for Theorem 3.3 uses a bit of geometry, and will be split between the following two questions. Assume we name the parabola E.

Question 3.2. (3 pts) Consider our fold that places P on line L. Let P' be the point on L to which P is folded to, and call the crease line C. Then, let X be the point on C such that $\overline{XP'}$ is perpendicular to L. Show that X is a point on our parabola E.

Solution: The key observation here is that folding will essentially reflect one side of the paper across the crease line C.

Since X is a point on C, we have that $\overline{XP}$ will reflect onto $\overline{XP'}$ when our fold is made. This shows that the length of $\overline{XP}$ is equal to the length of $\overline{XP'}$, so X is therefore a point on our parabola with focus P and directrix L.

Question 3.3. (3 pts) Next, show that this point X is the only point on both our crease line C and our parabola E, thus showing that C is tangent to E.

Solution: If we pick another point $Q \neq X$ on C, we see that using the same logic as the previous question, $\overline{QP} = \overline{QP'}$. However, since Q is not the same point as X, QP' is not perpendicular to our directrix L. Therefore, QP' is not the distance from L to Q meaning that the distance between the focus and Q is not equal to the distance between L and Q. What this shows is that X is the only point on our crease line C that's also on the parabola, thus meaning that X is tangent to this parabola.

Now that we have this theorem in our toolkit, we can observe a corresponding, tangible application by attempting to find the real roots of $f(x) = x^2 + ax + b$, where a and b are rational numbers.

Question 3.4. (2 pts) Compute the focus and directrix of the parabola $y = x^2 + ax + b$.

Solution: By completing the square, we can reshape this equation to our desired form of $(x-h)^2 = 4p(y-k)$:

$$y = x^2 + ax + b$$
$$\Rightarrow y - b = \left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}$$
$$\Rightarrow y - b + \frac{a^2}{4} = \left(x + \frac{a}{2}\right)^2$$
$$\Rightarrow 4 \cdot \frac{1}{4}\left(y - \left(b - \frac{a^2}{4}\right)\right) = \left(x - \left(-\frac{a}{2}\right)\right)^2$$

From this form, we see that our focus is $\left(-\frac{a}{2}, b - \frac{a^2}{4} + \frac{1}{4}\right)$, and our directrix is $y = b - \frac{a^2}{4} - \frac{1}{4}$.

Call the focus for the above parabola P and the directrix L, where L is of the form $y = k$ for some constant k. Using similar logic from above, we can fold P to some arbitrary point (t, k) on L. This ends up creating a crease line with the following equation, derived through calculus:

$$y = (2t + a)x - t^2 + b.$$

The values of t that allow our crease line to be tangent to the parabola at a root are, assuming both roots are real, $t = \frac{-a+\sqrt{a^2-4b}}{2}$ and $t = \frac{-a-\sqrt{a^2-4b}}{2}$.

Substituting one of those values of t into our crease equation results in the following messy equation:

$$y = x\sqrt{a^2 - 4b} + \frac{a}{2}\sqrt{a^2 - 4b} + 2b - \frac{a^2}{2}.$$

Question 3.5. (5 pts) Our final step is to find a point P' on our crease line that is easy to construct from the coefficients of our original quadratic. Find this point P', and then explain how the root can be found using P' (and other previous information). Assume that the x-axis is an already-constructed line. *Hint: what value of x would drastically simplify the above line equation?*

Solution: Notice that if we let $x = -\frac{a}{2}$, some cancellation occurs in the equation of our crease line, giving $y = 2b - \frac{a^2}{2}$. This gives us the point $P_2 = \left(-\frac{a}{2}, 2b - \frac{a^2}{2}\right)$, which is constructible from our original quadratic coefficients (and an origin + unit length), since the four basic arithmetic operations are fairly simple to do with origami. So, we can now fold P_1 onto L, and make the crease go through P_2 (using fold **(E)**). By construction, this crease corresponds to one of the t values that is tangent to the parabola at a root, so the intersection of the crease and x-axis must be a root.

Here's a less algebra-heavy way to solve for the real roots of $x^2 + ax + b = 0$, where a and b are rationals, known as **Lill's construction**:

1. On a graph, construct $A = (0, 1)$, and $B = (-a, b)$.
2. Draw a circle with diameter $\overline{AB}$ centered at the midpoint, C, of $\overline{AB}$.
3. If M and N are the two points where this circle intersects the x-axis, then assuming our origin is O, the x-coordinates of M and N, should they exist, will be solutions to $x^2 + ax + b = 0$.

Question 3.6. (3 pts) Show how the points C, M, and N can be constructed through folding. Assume that the y-axis and x-axis have been constructed already, and that M and N are both real points.

Solution: Point C can be obtained by first making a fold along the segment $\overline{AB}$ [fold **(A)**], and then folding A to B [(fold **(B)**], and the intersection of these two folds is point C.

Point M can be constructed by folding A on the x-axis and making sure this crease passes through our point C. [fold **(E)**]. Since N is assumed to be a real point, we can use another fold of type **(E)** along a different crease to get N.

Question 3.7. (4 pts) Show that the x-coordinates of M and N correspond to the roots of the quadratic equation $x^2 + ax + b = 0$.

Solution: The equation for our circle with center C is

$$\left(x + \frac{a}{2}\right)^2 + \left(y - \frac{1+b}{2}\right)^2 = r^2$$

Since we want to find the x-intercepts, set $y = 0$:

$$\left(x + \frac{a}{2}\right)^2 + \left(-\frac{1+b}{2}\right)^2 = r^2$$

Now, we substitute $r = \frac{1}{2}\sqrt{a^2 + (b-1)^2}$, as that is the radius of our circle.

$$\left(x + \frac{a}{2}\right)^2 + \left(-\frac{1+b}{2}\right)^2 = \frac{1}{4}(a^2 + (b-1)^2)$$

Simplifying and subtracting unneeded terms of a^2 and b^2 gives us

$$x^2 + ax + b = 0$$

which means that the x-coordinates of M and N are indeed the roots of the quadratic equation $x^2 + ax + b = 0$.

We have seen that origami, like straightedge and compass constructions, can solve quadratic equations. However, origami is more powerful than that, as it can also construct arbitrary cube roots. One way to do this is through the Beloch square.

Definition 3.4. Given two points, A and B, and two lines, r and s, the **Beloch square** is the square $WXYZ$ such that the two adjacent corners X and Y lie on r and s, respectively, and the sides $\overline{WX}$ and $\overline{YZ}$, or their extensions, pass through A and B, respectively.

Before we get into the Beloch square, however, we can first dive into the Beloch fold.

Definition 3.5. Given two points, A and B, and two lines, r and s, the **Beloch fold** is the single fold that places A onto r and B onto s simultaneously.

Question 3.8. (4 pts) Make a connection between the Beloch fold and Theorem 3.3. In terms of parabolas, what is the Beloch fold really doing?

Solution: From Theorem 3.3, we saw that if we fold a point P to a line l, the resulting crease line will be tangent to the parabola with focus P and directrix l.

The Beloch fold folds two points to two separate lines: folding A to r will make the crease be tangent to the parabola with focus A and directrix r, and folding B to s will make the crease be tangent to the B-focused and s-directrixed parabola. So, the resulting crease made by the Beloch fold is a common tangent to two parabolas.

Question 3.9. (8 pts) Given two points A and B and two lines r and s, detail a series of folding steps for constructing a Beloch square $WXYZ$ as detailed in Definition 3.4. Note: do not submit folded origami.

Solution: We first would like to compute the perpendicular distance, say x, from A to r, and then create a new line r' that is x distance away from r and also parallel to r, so that r is between A and r'. Do the same thing for B and s to create s'. Note that these lines can be made by folding over r and marking where A lands, folding the line that goes through both A and the marked point, and then making the fold that goes through the marked point and is also perpendicular to the previous fold. The same thing can be done with B, s, and s'.

Now, perform the Beloch fold that folds A onto r' and B onto s'. A will fold to a point A' on r', and B will fold to a point B' on s'. Note that the crease made in this fold will be the perpendicular bisectors of both $\overline{AA'}$ and $\overline{BB'}$. If we let X and Y be the midpoints of $\overline{AA'}$ and $\overline{BB'}$, respectively, then we see that X lies on r and Y lies on s. The segment $\overline{XY}$ is therefore one side of our Beloch square, and A and B are on opposite sides of this square.

Noting the distance between X and Y (which will be the length of each of the four sides of the square), we can then construct the Beloch square.

Question 3.10. (10 pts) Take r to be the y-axis, and take s to be the x-axis. Then, let $A = (-1, 0)$ and $B = (0, -2)$. If r' and s' are constructed to be the lines $x = 1$ and $y = 2$, detail a series of folds that ends up allowing us to construct the cube root of two within this setup.

Solution: As in Question 3.9, we construct lines r' and s' to be $x = 1$ and $y = 2$, respectively. If we use the Beloch fold to fold A onto r' and B onto s', this will make a crease that crosses r at a point X and crosses s at a point Y. Denoting O as the origin, we notice that right triangles $\triangle OAX$, $\triangle OXY$, and $\triangle OYB$ are all similar to each other due to the fact that $\overline{XY}$ is perpendicular to both $\overline{AA'}$ and $\overline{BB'}$.

From these similar triangles, we can construct the following equalities:

$$\frac{OX}{OA} = \frac{OY}{OX} = \frac{OB}{OY}$$

We can plug in $OA = 1$ and $OB = 2$, and notice that if we want to compute OX^3, we get that $OX^3 = OX \cdot \frac{OY}{OX} \cdot \frac{2}{OY} = 2 \Rightarrow OX = \sqrt[3]{2}$. This shows that we have constructed the cube root of two as OX.

In addition to constructing cube roots, Beloch's square also gives way to constructing solutions to cubic equations. Check out Lill's Method for further reading on this idea!

Theorem 3.6. For real numbers a, b, and c, if r is a real solution to $x^3 + ax^2 + bx + c = 0$, then given $(0, a)$, $(0, b)$, and $(0, c)$, it is possible to construct $(0, r)$ by folding.

4 The Bounds of Foldability (48 pts)

In this last chapter, we'll be exploring the bounds of what is and isn't foldable. Before we get started on looking into what types of shapes we're able to fold, we first look into what types of points we can construct, also known as the **origami numbers**.

To do this, we'll be visualizing an **infinitely large** sheet of paper, in all directions. After drawing a horizontal and vertical axis, we mark the points $(0, 0)$ and $(1, 0)$ on this sheet of paper. When we fold and unfold this paper (using straightedges), it will leave a crease which acts as a line. Another point in our paper will exist if it lies at the intersection of two formed creases.

Definition 4.1. A point (x, y) is **origami-constructible** if, starting with our infinitely large paper with $(0, 0)$ and $(1, 0)$ marked, we can make a series of folds so that two lines intersect at (x, y). Also, if the image of an origami constructible point P after getting reflected over a constructed line l is P', then P' is an origami constructible point.

With a set of known origami-constructible points, we can create more origami constructible points. For instance, suppose $(a, 0)$ and $(0, b)$ are both origami constructible points. To show that (a, b) is an origami constructible point, we can make a fold parallel to the x-axis going through $(0, b)$, and then make a fold parallel to the y-axis going through $(0, b)$. These two folds will intersect at (a, b).

Assume that every fold you make gets unfolded right afterwards (but the crease from that fold still remains).

Question 4.1. (3 pts) Suppose that (a, b) is an origami-constructible point. Explain how $(-a, -b)$ is an origami constructible point by detailing a series of folds that leads to $(-a, -b)$ being constructed.
Solution: Make a fold parallel to the x-axis, and the image of (a, b) then becomes $(a, -b)$. Then, make a fold parallel to the y-axis, and the image of $(a, -b)$ then becomes $(-a, -b)$, so $(-a, -b)$ is therefore an origami-constructible point.

Question 4.2. (3 pts) Suppose that (a, b) and (c, d) are both origami-constructible points. Explain how $(a + c, b + d)$ is an origami-constructible point by detailing a series of folds that leads to $(a + c, b + d)$ being constructed. You may assume that the origin and these two points are **not** collinear.
Solution: We can first make the fold, F_1, that goes through the origin and (a, b), and then the fold, F_2, that goes through the origin and (c, d). Then, make a fold, F_3, that's perpendicular to F_2 and goes through the point (a, b), and then make a fold F_4 that is perpendicular to F_3 and intersects F_3. For our last two folds, repeat the last two folds, but switching the roles of the two points. The end result is a parallelogram, with the corner being the point $(a + c, b + d)$.

So far, we've seen how one can find more origami-constructible points from previously discovered ones through addition and additive inverses. The same can be done through the use of multiplication and multiplicative inverses.

Question 4.3. (3 pts) Suppose for $a \neq 0$ that we have the origami-constructible point $(a, 0)$. Explain how $(1, \frac{1}{a})$ is an origami-constructible point by detailing a series of folds that creates $(1, \frac{1}{a})$.

Solution: Fold a line perpendicular to the x-axis and intersecting $(a, 0)$, and then fold a line perpendicular to the y-axis and intersecting $(0, 1)$. This creates the point $(a, 1)$. Fold the line intersecting the origin and $(a, 1)$. Then, fold the line perpendicular to the x-axis and intersecting $(1, 0)$. The point of intersection is $\left(1, \frac{1}{a}\right)$.

Question 4.4. (3 pts) Suppose we have the origami-constructible points $(0, a)$ and $(b, 0)$. Explain how (b, ab) is an origami-constructible point by detailing a series of folds that creates (b, ab).

Solution: Construct a vertical line through $(1, 0)$ and a horizontal line through $(0, a)$. This constructs the point $(1, a)$, so after making the fold intersecting the origin and $(1, a)$, the slope for this fold is a. Finally, make a vertical fold intersecting $(b, 0)$, and so the intersection between this fold and the previous fold is (b, ab).

Instead of thinking of our points as being in $\mathbb{R}^2$ space, we can also embed our origami-constructible points into the complex plane. With all of these addition and multiplication rules being satisfied, we can now state that the set of all origami-constructible points $\mathscr{O}$ is a **subfield** of $\mathbb{C}$, the set of all complex numbers $a + bi$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.
Roughly speaking, this means that with the binary operators of adding and multiplying, the set of all origami-constructible points satisfies certain properties, including but not limited to:

- Closure: adding any two origami numbers and multiplying any two origami numbers both result in an origami number.
- Distributivity: If $a, b, c \in \mathscr{O}$, then $a * (b + c) = a * b + a * c$.
- Existence of identity elements: There exists some $e_1 \in \mathscr{O}$ where $b + e_1 = b$ for any $b \in \mathscr{O}$. Also, there exists some e_2 where $b * e_2 = b$ for any $b \in \mathscr{O}$.
- Existence of inverse elements: There exists some $i_1 \in \mathscr{O}$ where $b + i_1 = 0$ for any $b \in \mathscr{O}$. Also, there exists some i_2 where $b * i_2 = 1$ for any nonzero $b \in \mathscr{O}$.

This gives our set of origami numbers some added structure, allowing us to make more generalized claims about our set. Some other examples of fields are the rational numbers $\mathbb{Q}$ (the set of all numbers $\frac{a}{b}$, where a and b are integers and b is not equal to 0), and the complex numbers (the set of all numbers $a + bi$, where a and b are both real numbers, and $i = \sqrt{-1}$).

This definition of a field also allows us to define an order on fields.

Definition 4.2. A **field extension** K/F occurs when we have two fields, say F and K, where $F \subset K$ (meaning all of F is contained within K), and K contains F as a subfield.

Some examples of this are $\mathbb{R}/\mathbb{Q}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$, where $\mathbb{Q}(\sqrt{2})$ is the set of all numbers $a + b\sqrt{2}$, where a and b are rationals.

We can interpret $K = F(\alpha)$ as K being the smallest field that contains both the field F and the element α.

Definition 4.3. The **minimal polynomial** of α is the unique polynomial $f(x)$ with coefficients in F of smallest degree such that $f(\alpha) = 0$ and the leading coefficient is 1.

For example, suppose $F = \mathbb{Q}$ and $\alpha = \sqrt{2}$. The minimal polynomial of $\alpha = \sqrt{2}$ over the rationals is $f(x) = x^2 - 2$, because $\sqrt{2}$ satisfies this polynomial, and $x^2 - 2$ is irreducible over the rationals (we cannot factor this polynomial into $(x - \sqrt{2})(x + \sqrt{2})$ because $\sqrt{2}$ is not rational).

Question 4.5. (6 pts) Find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over the rational numbers $\mathbb{Q}$, and prove that this polynomial is indeed minimal.
Solution: Set $\alpha = \sqrt{2}+\sqrt{3}$. Squaring both sides yields $\alpha^2 = 5+2\sqrt{6}$. Then, subtract 5 from both sides and square both sides again to get $\alpha^4 - 10\alpha^2 + 1 = 0$, which is a polynomial over the rationals with $\sqrt{2}+\sqrt{3}$ as a root.
We now show that this polynomial is minimal by proving it is irreducible over the rationals. For one, there are no rational roots to this polynomial by the rational root theorem. Then, by attempting to factor it into two quadratic polynomials $(x^2+ax+b)(x^2+cx+d)$, we get a system of equations $a+c=0, ac+b+d=-10, ad+bc=0, bd=1$ which is not solvable over rationals either. If we try we get $a=-c, a(b-d)=0$ from equations 1 and 3. We try both cases: if $a=0$, we have $b+d=-10, bd=1$, which implies $b^2+10b+1$ has rational roots (which it doesn't by rational root theorem.) If $b-d=0$, then $b=d$ and $bd=1$ implies $b=d=\pm 1$. This gives $-a^2=-8,-12$ both of which do not have rational solutions. Therefore, this polynomial can't be factored over the rationals and is minimal.

Lemma 4.4. If p is a prime number and ω is a root (not equal to 1) of $x^p - 1 = 0$, then the minimal polynomial of ω is $x^{p-1} + x^{p-2} + \cdots + 1$.

We also define a bit more related machinery.

Definition 4.5. The **degree** of the field extension $K = F(\alpha)$ of F, denoted as $[K : F]$, is the degree of a minimal polynomial f with coefficients in F and $f(\alpha) = 0$.

Definition 4.6. A **2-3 tower** is a nested sequence of fields

$$\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{C}$$

such that $[F_i : F_{i-1}] = 2$ or 3 for all $1 \le i \le n$.

We begin with our given points $(0,0)$ and $(1,0)$ (and our given line, the horizontal axis).
We have shown that we can use our origami folding operations to construct any point in the complex plane with rational coordinates ($a+bi$ for all $a,b \in \mathbb{Q}$, and $i=\sqrt{-1}$). This can be defined as the field extension $\mathbb{Q}(i)$.

With only this extension, our "tower" is $\mathbb{Q} \subset \mathbb{Q}(i)$, and $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ because x^2+1 is the minimal polynomial of i over $\mathbb{Q}$.
Constructing $\sqrt[3]{2}$ yields us the 2-3 tower of $\mathbb{Q} \subset \mathbb{Q}(i) \subset \mathbb{Q}(i, \sqrt[3]{2})$, all of which are a subset of $\mathscr{O}$.

Question 4.6. (10 pts) Prove that if there exists a 2-3 tower

$$\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{C}$$

such that $\alpha \in F_n$, then $\alpha \in \mathscr{O}$.

Solution: We make use of the fact that by Theorem 3.6, we can use origami to solve all real solutions for quadratics and cubics. This means that any 2-3 tower of fields will correspond to fields that have minimal polynomials that are quadratic or cubic, meaning we can factor these polynomials completely in $\mathscr{O}$.

Firstly, let the following be a 2-3 tower with $\alpha \in F_n$

$$\mathbb{Q} = F_0 \subset \cdots \subset F_n \subset \mathbb{C}$$

We now proceed by induction. When $n = 0$, we see that $F_0 = \mathbb{Q}$, and since all rationals are origami numbers, we have shown that our base case is true.

Now assume that $F_{n-1} \subset \mathscr{O}$ by our induction hypothesis. Let $\alpha \in F_n$, and f be the minimal polynomial of α over F_{n-1}. This minimal polynomial will be of degree 3 or less, since $[F_n : F_{n-1}] = 2$ or 3. If f had degree 1, then this would make the extension F_n redundant, meaning $\alpha \in \mathscr{O}$. However, if f had degree 2 or 3, then all the roots of f can be constructed by the quadratic formula or even Cardano's formula, both of which use only square and cube roots, which are origami constructible. This means that the minimal polynomial can indeed be factored into a product of linear terms $(x - r_1)(x - r_2)(x - r_3)$. Therefore, α is indeed an origami number.

Finally, we extend this idea of foldability beyond origami-constructible points, and into the realm of n-gons. With the information we went over, we can make a bold claim about which n-gons can be constructed by origami. However, we present some terminology so that we're better equipped for our final big theorem.

Definition 4.7. A **splitting field** of a polynomial $f(x)$ over a field F is the smallest field extension K of F in which the polynomial splits completely into linear factors. In other words, it is the smallest field containing F and all the roots of $f(x)$.

Theorem 4.8. Let $\alpha \in \mathbb{C}$ be a solution to a polynomial with coefficients in $\mathbb{Q}$, and let L be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. Then, α is an origami number if and only if $[L : \mathbb{Q}] = 2^a 3^b$ for integers $a, b \geq 0$.

Definition 4.9. A prime p is called a **Pierpont prime** if $p > 3$ and p is of the form $2^a 3^b + 1$ for integers $a, b \geq 0$.

Question 4.7. (20 pts) Prove that a regular n-gon can be constructed by origami if and only if $n = 2^a 3^b p_1 p_2 \dots p_k$ for some integers $a, b \geq 0$ and where $p_1, \dots, p_k$ are distinct Pierpont primes.

Solution: We first notice that we can construct an n-gon if and only if we can construct the nth roots of unity in $\mathbb{C}$, one of them being $\zeta_n = e^{2\pi i/n}$. From that, we have that $\mathbb{Q}(\zeta_n)$ is the splitting field of the polynomial $z^n - 1$ in the rationals. By Theorem 4.8, ζ_n is origami constructible if and only if $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = 2^a 3^b$ for some $a, b \geq 0$.

The minimal polynomial of ζ_n over the rationals is

$$\phi_n(z) = \prod_{\substack{0 \leq i < n \\ \gcd(i,n)=1}} (z - \zeta_n^i)$$

usually denoted as the nth cyclotomic polynomial. This tells us that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \deg(\phi_n(z)) = \phi(n)$, where $\phi(n)$ is the number of positive integers i less than n with $\gcd(i, n) = 1$.

A useful result from this function is that $\phi(n) = n \prod_{p|n}(1 - 1/p)$ for integers $n > 1$, and this is taken over all primes p that divide n. We are now ready to tackle the proof.

($\implies$) Suppose that $n = 2^a 3^b p_1 \dots p_k$, where $a, b \geq 0$, and all the primes are Pierpont primes. Using $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) = n \prod_{p|n}(1 - 1/p)$, we get that this will equal one of four different cases, based on whether or not a or b are zero or positive:

- If a and b are greater than 0, then $\phi(n) = 2^a 3^{b-1}(p_1 - 1) \dots (p_k - 1)$
- If $a > 0$ and $b = 0$, then $\phi(n) = 2^{a-1}(p_1 - 1) \dots (p_k - 1)$
- If $a = 0$ and $b > 0$, then $\phi(n) = 2 \cdot 3^{b-1}(p_1 - 1) \dots (p_k - 1)$
- If $a = b = 0$, then $\phi(n) = (p_1 - 1) \dots (p_k - 1)$

Notice that all of these are a power of 2 times a power of 3 since all the primes are Pierpont primes. Therefore, using Theorem 4.8, we can say that ζ_n is indeed an origami number.

($\impliedby$) Conversely, suppose ζ_n is an origami number. We still have that $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n) = n \prod_{p|n}(1 - 1/p)$, but now suppose that we can write n into its prime factorization: $n = q_1^{a_1} \dots q_r^{a_r}$. This means that

$$\phi(n) = q_1^{a_1 - 1}(q_1 - 1) \dots q_r^{a_r - 1}(q_r - 1)$$

If some q_i was odd, then that would mean that $q_i = 3$, or $a_i = 1$, or we would violate the condition that the index of the splitting field over $\mathbb{Q}$ is $2^a 3^b$. It's also worth noting that every $(q_i - 1)$ is therefore necessarily a power of 2 times a power of 3, thus making every q_i a Pierpont prime. Therefore, every prime factor of n is either a 2, a 3, or a Pierpont prime.

As a result of this theorem, we see that the undecagon (11-gon) is the smallest regular polygon that is not constructible by straight-crease, single-fold origami!