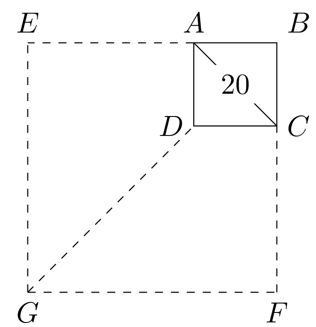
1. The area of square $EBFG$ is 9 times the area of square $ABCD$, as shown in the diagram below. The diagonal $\overline{AC}$ of $ABCD$ has length 20. Compute the length of $\overline{DG}$.



Answer: 40

Solution:

Since $ABCD$ and $EBFG$ are squares whose areas are in the ratio $9 : 1$, their sides are in the ratio $\sqrt{9} : 1 = 3 : 1$. Triangles $\triangle ACD$ and $\triangle BGF$ are both 45-45-90 triangles, so they are similar with ratio $CD : FG = 1 : 3$. Thus, $BG = 3AC = 3 \cdot 20 = 60$. Also, by symmetry $BD = AC = 20$. We have $BG = BD + DG = 60 = 20 + DG$ so $DG = \boxed{40}$.

2. Wen writes a positive integer W on the board. Repeatedly, she multiplies this integer by 2, writes the result on the board, and erases the original number. At some point, the value written on the board is 2024. Compute the smallest possible value of W.

Answer: 253

Solution 1: This problem is equivalent to dividing 2024 by 2 until it is no longer possible. We have $\frac{2024}{2} = 1012$, then $\frac{1012}{2} = 506$, then $\frac{506}{2} = 253$. Since this is not even, we are done. The smallest possible integer she could have started with is $\boxed{253}$.

Solution 2: We can write the prime factorization of $2024 = 2^3 \cdot 11 \cdot 23$. To minimize W, we should remove all factors of 2. We can pull out three factors of 2 and we will be left with $23 \cdot 11 = \boxed{253}$, which is the least possible number Wen could have started with.

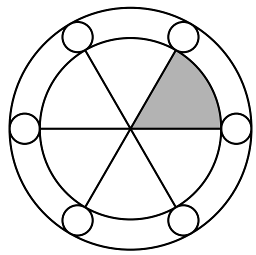
3. Compute the third largest factor of the third largest factor of the third smallest positive integer whose third largest factor has at least three factors. Recall that every positive integer is a factor of itself.

Answer: 2

Solution: Let n be the third smallest positive integer whose third largest factor has at least three factors. We want to compute the third largest factor of the third largest factor of n. First, we need n to have a factor that has at least 3 factors. Some small numbers with at least 3 factors are 4 (1, 2, 4) and 6 (1, 2, 3, 6) and 8 (1, 2, 4, 8). Therefore the third largest factor of n will probably be one of these numbers.

By counting up, we see that the smallest positive integer whose third largest factor has three factors is 12, whose third largest factor is 4. The next candidate is 16, whose third largest factor is also 4. Counting up a little more shows that 18 is the third smallest candidate, meaning $n = 18$ with factors $1, 2, 3, 6, 9, 18$, and third largest factor 6. Then the third largest factor of 6 is $\boxed{2}$.

4. Jonathan is riding a unicycle. The unicycle wheel has a large outer circle, 6 small circles, and a medium inner circle divided into 6 congruent sectors by 3 spokes, as shown in the diagram below (diagram not to scale). The smallest circles have radius 1, the largest circle has radius 7. All six small circles are tangent to the inner circle at an endpoint of a spoke, and tangent to the outer circle. Compute the area of the shaded sector.



Answer: $\frac{25\pi}{6}$

Solution:

Let the radius of the inner medium circle be r. Connecting the center of the large circle to a point of tangency with a smaller circle, we have $7 = r + 2 \cdot 1$ so $r = 5$. The sectors divide the area of the medium circle in 6, so the area of the shaded sector is $\frac{1}{6}\left(\pi \cdot 5^2\right) = \boxed{\frac{25\pi}{6}}$.

5. Compute the smallest three-digit positive integer with distinct nonzero digits satisfying the property that it is not divisible by any of its digits. For example, 426 does not have this property because it is divisible by 2 and 6.

Answer: 239

Solution: Let this number be x. We first note that x cannot include any 1 in its digits as 1 divides every three digit number. Therefore, because x has all distinct and nonzero digits, it must be at least 234. We then try numbers starting from 234; we could try all numbers, but to speed things up we note x cannot be even, otherwise it would be divisible by its first digit, 2. Therefore, we try odd numbers:

- First, 235 is divisible by 5, one of its digits.
- Next, 237 is divisible by 3, one of its digits.
- Now, 239 is not divisible by $2, 3$, or 9. Therefore, our answer is $\boxed{239}$.

6. Compute $(x + y)^{100}$ given

$$\left(\frac{x^2 + 2xy + y^2}{y + z}\right)^{100} = 1, \quad \left(\frac{y^2 + 2yz + z^2}{x + z}\right)^{50} = 8, \quad \left(\frac{x^2 + 2xz + z^2}{x + y}\right)^{25} = 16.$$

Answer: 16

Solution:

Seeing expressions of the form $a^2 + 2ab + b^2$ we are inspired to factor them as $(a + b)^2$:

$$\left(\frac{(x + y)^2}{y + z}\right)^{100} = 1, \quad \left(\frac{(y + z)^2}{x + z}\right)^{50} = 8, \quad \left(\frac{(x + z)^2}{x + y}\right)^{25} = 16.$$

Considering exponent rules, we can expand these equations out to

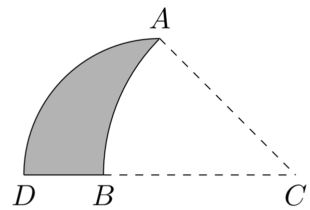
$$\frac{(x+y)^{200}}{(y+z)^{100}} = 1, \quad \frac{(y+z)^{100}}{(x+z)^{50}} = 8, \quad \frac{(x+z)^{50}}{(x+y)^{25}} = 16.$$

Taking the product of all of these equations, we see that the powers of $(y+z)$ and $(x+z)$ will cancel out and we will get

$$(x+y)^{175} = 8 \cdot 16 = 2^7$$

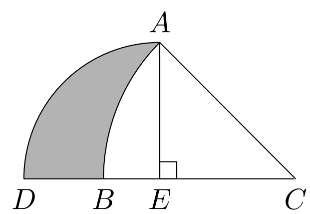
Finally, note that $175 = 25 \cdot 7$, so we can take a 7th root and get $(x+y)^{25} = 2$, and then raising this to the 4th power gives $(x+y)^{100} = 2^4 = \boxed{16}$.

7. During his escape from Alcatraz Island, Aditya swims to San Francisco and sees a shark fin above the water, indicated by the shaded area in the diagram below. The shark fin is formed by a quarter circle of radius 3 with arc $\widehat{AD}$ cut by its overlap with a 45° sector of the circle centered at C passing through A and B (on the two straight sides of the quarter circle). Compute the area of the shark fin.



Answer: $\mathbf{\frac{9}{2}}$

Solution:



Dropping the perpendicular from A to $\overline{CD}$ (intersecting at E), we get that the area of the shark fin is equal to the area of the shape enclosed by $\overline{CD}, \overline{AC}, \widehat{AD}$ (call this ACD) minus the area of the sector of the circle centered at C. The area of ACD is then equal to the sum of the area of the quarter circle of radius 3 plus the area of the triangle with base CE and height AE. The quarter circle has radius 3, so its area is $\frac{\pi \cdot 3^2}{4} = \frac{9\pi}{4}$.

Next, we find the area of the circle sector with center C. Since the quarter circle has radius 3, we know that $AE = 3$. Then, we have that $\triangle ACE$ is a right isosceles triangle with right angle at E due to the 45° angle. This means $AE = CE = 3$, and $AC = 3\sqrt{2}$. So, the radius of the circle sector centered at C is $3\sqrt{2}$, giving the area of the sector as $\frac{\pi \cdot (3\sqrt{2})^2}{8} = \frac{9\pi}{4}$.

Thus, the area of the shaded region is

$$\frac{9\pi}{4} + \frac{3 \cdot 3}{2} - \frac{9\pi}{4} = \boxed{\frac{9}{2}}.$$

8. Isaac has a steel tube that is 2024 units long. He wants to cut this tube into C smaller pieces such that no three pieces can be the sides of a triangle with positive area, and each piece has a unique positive integer length. Compute the greatest possible value of C.

Answer: 14

Solution: We start building a list of integers that satisfies this property, and stop when the sum of our list exceeds 2024. The first three integers are $1, 2, 3$. Then, to ensure that we can't form any triangles, we must prevent the triangle inequality from being satisfied: we must force $a + b \leq c$ for all ordered triples (a, b, c). To enforce this, the next number we add should be the sum of the two largest elements of our list: in this case, 5. The sequence is then $1, 2, 3, 5, 8, 13, \ldots$ up until the 14th term, 610. The sum of those 14 elements is 1595, which leaves a remainder of 2024-1595 = 619, which is not enough to add the next term 987. Therefore, if we were to cut the tube into 15 pieces $1, 2, 3, 5, \ldots, 377, 610, 619$, we can see that the final triple of integers does satisfy the triangle inequality and would form a triangle with positive area. To prevent this, we add the remaining length to the last integer 610 giving the best possible list of $\boxed{14}$ integers.

9. Let $\log_2^*(n)$ be the number of times we need to apply $\log_2$ to n to get a number less than 1. This function grows very slowly, but a useful application is to extended exponentiation:

   Let $a \uparrow^1 b = a^b$, and $a \uparrow^{n+1} b = \underbrace{a \uparrow^n (a \uparrow^n (\cdots (a \uparrow^n a)))}_{b-1 \quad \uparrow^n\text{'s}}$. For instance, $4 \uparrow^2 3 = 4 \uparrow^1 (4 \uparrow^1 4)$.

   Compute $\log_2^*(2 \uparrow^3 4)$.

   Answer: 65537 OR $\mathbf{2^{16} + 1}$

   Solution: We'll expand $2 \uparrow^3 4$ to get $2 \uparrow^2 (2 \uparrow^2 (2 \uparrow^2 2))$, which gives $2 \uparrow^2 (2 \uparrow^2 4)$, which is a power tower of size $2 \uparrow^2 4 = 65536$.

   One thing to notice is that $\log_2(2^n) = n$, so every successive logarithm will cut off a layer of the tower. We need $2 \uparrow^2 4$ applications of $\log_2$ to cut the tower down to 1, and then one more application to yield a number less than 1. Thus $\log_2^*(2 \uparrow^3 4) = 2^{16} + 1 = \boxed{65537}$.

10. Suppose N, a, and b are positive integers such that $N = a^3 + b^3 - a^2b - ab^2$. Given that N has exactly 6 factors, compute the least possible value of N.

    Answer: 32

    Solution: For brevity, let $f(a, b) = a^3 + b^3 - a^2b - ab^2$. We can factor: $a^3 + b^3 - a^2b - ab^2 = (a + b)(a - b)^2$. Since $6 = 6 \cdot 1 = 3 \cdot 2$, by the formula for the number of factors, $f(a, b)$ must either be a prime raised to the 5th power or in the form pq^2 where p and q are distinct primes.

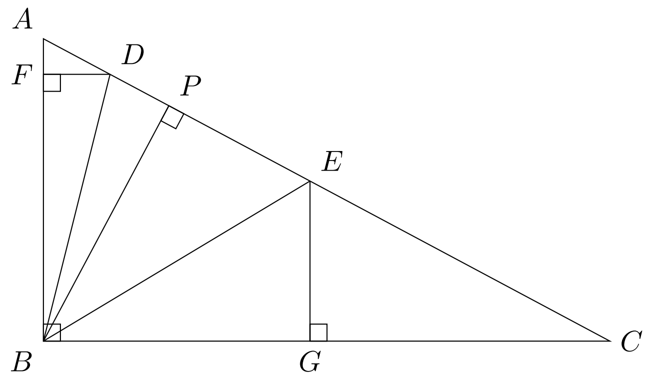
    If $f(a, b)$ is in the form pq^2, then $p = a + b$ and $q = a - b$. To minimize $f(a, b)$ while keeping both a and b positive integers, p and q should be the minimum primes with the same parity with $p > q$. Therefore, $p = 5$ and $q = 3$ will give the minimum value of 45 for this form.

    If $f(a, b) = p^5$, it could only be less than 45 if $p = 2$ where $f(a, b) = 32$. Then $(a+b)(a-b)^2 = 2^5$. Since a and b are integers and $a + b > a - b$, we need $a + b$ and $a - b$ to be powers of 2 where $a + b$ is a larger power of 2 than $a - b$. If $a + b = 2^m$ and $a - b = 2^n$, we need $m + 2n = 5$ where $1 \leq n \leq m$ (to ensure that $a + b$ and $a - b$ have the same parity and the system has integer solutions). The only possibility is $m = 3, n = 1$, which gives us $a + b = 8$ and $a - b = 2$. Solving for the system of equations gives $a = 5$ and $b = 3$, so $N = 32$ is achievable. Therefore, $\boxed{32}$ is the minimum value.

11. Let $\triangle ABC$ be a right triangle such that $\angle B = 90^\circ$. Points D and E are placed on $\overline{AC}$ such that $AB = AE$ and $BC = DC$. Given that $AD = 2$ and $EC = 9$, compute $BD \cdot BE$.

    Answer: $\mathbf{\frac{720\sqrt{2}}{17}}$

    Solution 1:



Let $DE = x$, in right triangle $\triangle ABC$, $AB = 2+x$, $BC = 9+x$, $AC = 2+x+9 = 11+x$. Using Pythagorean Theorem, $(2+x)^2+(9+x)^2 = (11+x)^2$, and we get $x = \pm 6$. Thus, $DE = 6$, right triangle $\triangle ABC$ has side lengths 8, 15, 17. Drop an altitude from point B onto $\overline{AC}$ and label this point P. Similarly, drop an altitude from D onto $\overline{AB}$ and label this point F, and then drop an altitude from E onto $\overline{BC}$ and label this point G.

From here, we do some angle chasing. Let $\angle EBC = \alpha$, since $\triangle ABC$ is a right triangle, $\angle ABC = 90^\circ$. Because $\triangle ABE$ is isosceles with $AB = AE$, $\angle ABE = \angle AEB = 90^\circ - \alpha$, $\angle A = 2\alpha$. $\angle ABE = 90^\circ - \alpha$, $\angle PBE = 90^\circ - (90^\circ - 2\alpha) - \alpha = \alpha$. Therefore, $\triangle BPE \sim \triangle BGE$ by AA similarity, and further $\triangle BPE \cong \triangle BGE$ since they share the same hypotenuse. Then $\angle ACB = 90^\circ - 2\alpha$, with $\triangle CBD$ isosceles so we have $\angle BDC = \angle DBC = 45^\circ + \alpha$. Thus, $\angle ABD = \angle DBP = 45^\circ - \alpha$. By the same argument as before, we also see that $\triangle BFD \cong \triangle BPD$.

From here, we can compute all of the lengths we want. In right triangle $\triangle ABC$, $BF = BP = \frac{8\cdot 15}{17} = \frac{120}{17}$ by the fact that $BP \cdot AC = AB \cdot BC$. Since $\angle AFD = 90^\circ$, $\triangle AFD \sim \triangle ABC$ and $FD = AD \cdot \frac{BC}{AC} = 2 \cdot \frac{15}{17} = \frac{30}{17}$. This allows us to compute

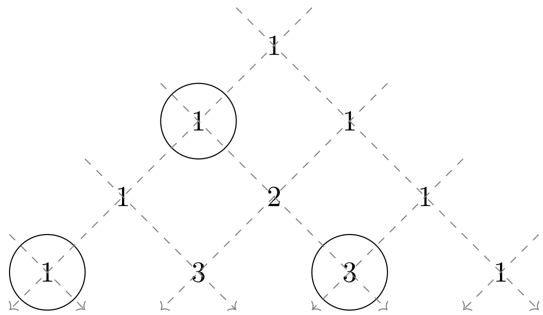
$$PE = AB - AD - DP = AB - AD - DF = 8 - 2 - \frac{30}{17} = \frac{72}{17}.$$

By Pythagorean Theorem, $BD = \sqrt{DP^2 + BP^2} = \sqrt{\left(\frac{30}{17}\right)^2 + \left(\frac{120}{17}\right)^2} = \frac{30\sqrt{17}}{17}$. With the same idea, $BE = \sqrt{PE^2 + BP^2} = \sqrt{\left(\frac{72}{17}\right)^2 + \left(\frac{120}{17}\right)^2} = \frac{24\sqrt{34}}{17}$. Finally, we have $BD \cdot BE = \frac{30\sqrt{17}}{17} \cdot \frac{24\sqrt{34}}{17} = \boxed{\frac{720\sqrt{2}}{17}}$.

Solution 2: Rotate $\triangle BAD$ counterclockwise around B until $\overline{BA}$ lies on $\overline{BC}$, and scale it with a factor of $\frac{15}{8}$ to get a new triangle $\triangle BCD'$. Then $\triangle BCD' \sim \triangle BAD$, $\frac{BC}{BA} = \frac{15}{8}$, $CD' = AD \cdot \frac{15}{8} = 2 \cdot \frac{15}{8} = \frac{15}{4}$. Noticing that $\angle BAC + \angle BCA = 90^\circ$, $\angle DCD' = \angle DCB + \angle BCD' = 90^\circ$. Thus, $\triangle DCD'$ is a right triangle. Similarly, $\angle DBD' = 90^\circ$. In right triangle $\triangle DBD'$, $\frac{BD}{BD'} = \frac{AB}{CB} = \frac{8}{15}$. And $DD' = \sqrt{DC^2 + CD'^2} = \sqrt{(6+9)^2 + (\frac{15}{4})^2} = \frac{15\sqrt{17}}{4}$. Thus, $DB = DD' \cdot \frac{8}{17} = \frac{30\sqrt{17}}{17}$. With the same idea, we rotate $\triangle BEC$, and we could get $BE = \frac{24\sqrt{34}}{17}$. Thus, $BD \cdot BE = \frac{30\sqrt{17}}{17} \cdot \frac{24\sqrt{34}}{17} = \boxed{\frac{720\sqrt{2}}{17}}$.

12. Define an *upright* triangle to be a set of three distinct vertices on Pascal's triangle, where two vertices are on the same row and the third vertex is above both points and shares a diagonal with

each of them. For example, the diagram below shows the first four rows of Pascal's triangle, and the three circled numbers are vertices of an upright triangle. Compute the sum of the vertices over all upright triangles on the first 10 rows of Pascal's triangle.



Answer: 9207

Solution: Each point on the first 10 rows of Pascal's triangle appears in 9 upright Pascal triangles (that are contained in the first 10 rows). To show this, for any point $\binom{a}{b}$ where $0 \le b \le a \le 9$, this point will be the top vertex of one triangle for each row that is beneath that vertex. There are a total of $9 - a$ triangles with a top vertex of $\binom{a}{b}$. This point will also be the bottom left vertex of one triangle for each point of the Pascal's triangle on the same row as $\binom{a}{b}$ that is right of it. There are $a - b$ such triangles (with right vertices $\binom{a}{b+1}, \binom{a}{b+2}, \cdots, \binom{a}{a}$). This point will also be the bottom right vertex of one triangle for each point of the Pascal's triangle on the same row as $\binom{a}{b}$ that is left of it. There are b such triangles (with left vertices $\binom{a}{0}, \binom{a}{1}, \cdots, \binom{a}{b-1}$). Therefore, the number of triangles that point $\binom{a}{b}$ is part of is $(9 - a) + (a - b) + (b) = 9$.

This means that the sum of the vertices over all upright equilateral triangles on the first 10 rows of the Pascal's Triangles is 9 times the sum of the first 10 rows of the Pascal's triangle. We know that row i has the sum 2^i (the relevant combinatorial identity can be proved by binomial theorem on $(1 + 1)^i$). By summing geometric series, the sum of the first 10 rows is $1 + 2 + 2^2 + \cdots + 2^9 = 2^{10} - 1$, making our answer $9(2^{10} - 1) = \boxed{9207}$.

13. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Additionally, let $\{x\}$ denote $x - \lfloor x \rfloor$. For example, $\lfloor \pi \rfloor = 3$ and $\{\pi\} = 0.1415\ldots$. Compute the integer n such that there are exactly 2024 positive solutions x to the equation

$$x^{\lfloor x \rfloor^{\{x\}}} = n.$$

Answer: 2028

Solution: Treating the LHS as a function $f(x) = x^{\lfloor x \rfloor^{\{x\}}}$, we notice that the behavior of this function is difficult to think about over a large interval, but over a smaller interval like $[3, 4)$ it may be easier. For x in this interval, we have $\lfloor x \rfloor = 3$, so our function over this interval is equivalent to $x^{3^{x-3}}$. This is a continuous, increasing function of x, which allows us to reason graphically about it. For $x = 3$ we have $f(x) = 3^{3^0} = 3$, and for x just under 4 we have $f(x) \approx 4^{3^1} = 64$. Thus, if $3 \le n < 64$ there is exactly one solution $f(x) = n$ on this interval, and otherwise there is no solution. We can generalize this to the conclusion that there is exactly one solution to $f(x) = n$ for $m \le x < m + 1$ if and only if $m \le n < (m + 1)^m$.

At this point, a reasonable approximation seems to be $n = 2024$ since the function grows very rapidly and there exists a solution to $f(x) = 2024$ in the interval $[2024, 2025)$ but no solution greater than 2025. However, the smallest solution occurs in the interval $[5, 6)$, which we can quickly see by approximating $5^4 < 2024$ and $6^5 > 2024$, therefore there are only $2024 - 5 + 1 = 2020$ solutions in the case $n = 2024$. Our next guess might then be $n = 2028$, since we need four more solutions. We don't "lose" solutions on the lower end because 6^5 is still much greater than 2028, but we get 4 more solutions on the upper end. In particular, there is exactly one solution in the interval $[i, i+1)$ for $i = 5, 6, \cdots, 2028$. Thus, the answer is $\boxed{2028}$.

14. For each positive integer n, let $s(n)$ be the sum of its digits and let $p(n)$ be the product of its digits. Compute the number of positive integers $n \leq 10^6$ that satisfy $s(n) - p(n) = 5$.

Answer: 285

Solution: There are two cases to consider. For now, we'll ignore the order of the digits: we can account for it later. First, suppose that the digit 0 does not appear in n. Then, consider the effect of adding any positive digit d to a number n. The effect on the sum of the digits is to add d, while the effect on the product of the digits is to multiply by d. If we start with a single digit number (which satisfies $s(n) = p(n)$), then we want to be able to add a digit so that $s(d) > p(d)$. However, notice that $x + d \leq dx$ for any $x, d \geq 2$. Therefore, we must add the digit 1 to increase $s(n)$ relative to $p(n)$ (the relative increase is just by 1). We can only have at most a 6 digit number, so we will have to add 5 1s to any single digit for this to be possible with no digits equal to 0.

This is both necessary and sufficient to satisfy $s(n) - p(n) = 5$, so now we just need to count how many ways there are to do this. If the last digit is 1, then we're just permuting 6 1s and there's only 1 way to do that. If the last digit is anywhere from 2 to 9, then we have 6 options of where to put it and then fill everything else with 1s, which gives us $6(8) = 48$ more possibilities, totalling to 49.

Now, suppose we do have 0 occur as a digit in n. Then the sum of the digits must just be 5, since the product also becomes 0. We can count the number of possibilities here with dots and dividers: there are 5 stars with 6 digits to go in, which gives us a total of $\binom{10}{5} = 252$ more numbers. However, we have to remove the cases where all the 0s occur as leading 0s, which means that n doesn't actually have any 0s in its digits. To do this, we can sum over the number of leading 0s that occur. If there are k leading 0s, then we can ignore the first k dividers, and give every remaining place a dot directly. This would give us $5 - k$ dividers and $5 - (6 - k) = k - 1$ dots, which has $\binom{4}{k-1}$ ways of being organized. We see that k can range from 1 to 5 (since we must have at least 1 zero and can't have all 6 be zero), which means we have the sum $\sum_{k=0}^{4} \binom{4}{k}$ which is just $2^4 = 16$. So, there are actually $252 - 16 = 236$ possible n with at least one digit being 0.

The total number of possible n is therefore $49 + 236 = \boxed{285}$.

15. Arjun considers the parabola described by $y = x^2 - 2x + 2$ in the coordinate plane and chooses some θ uniformly at random from the interval $[0, 2\pi)$. He then rotates the parabola about its vertex counterclockwise by θ and counts the number of times the resulting parabola intersects the coordinate axes. The probability that there are exactly four intersections can be expressed as $\frac{\pi + \arcsin(r) - \arccos(r)}{2\pi}$ for some real number r. Compute r.

Answer: $\sqrt{5} - 2$

Solution: A key observation is that after rotation, the parabola intersects the axes four times if and only if it intersects each axis at two distinct points. Next, note that $y = x^2 - 2x + 2$ is the translation of $y = x^2$ by one unit up and one unit to the right in the coordinate plane.

Rotating about the vertex of this parabola seems quite challenging, so it will likely be useful to translate the entire problem back so the parabola has vertex (0,0). Now, we are instead interested in rotating $y = x^2$ and counting intersections with $x = -1$ or $y = -1$. Still, rotating the parabola seems difficult. One idea is to use complex numbers to parameterize points on the parabola, as modeling rotation with complex numbers is very convenient. The parabola in the coordinate plane is the set of points (a, a^2) for real numbers a, which is equivalent to $a + a^2 i$ for real a. We can model rotation counterclockwise by θ as multiplication by the complex number $\cos(\theta)+i\sin(\theta)$, so the parabola after rotation by θ is the locus of $z = (a+a^2 i)(\cos(\theta)+i\sin(\theta)) = (a\cos(\theta) - a^2\sin(\theta)) + (a^2\cos(\theta) + a\sin(\theta))i$ as a varies over the reals. For this parabola to have two intersections with each of the lines $x = -1$ and $y = -1$, the equations

$$\mathrm{Re}(z) = a\cos(\theta) - a^2\sin(\theta) = -1$$

$$\mathrm{Im}(z) = a^2\cos(\theta) + a\sin(\theta) = -1$$

must each have two distinct solutions. These are quadratic equations in a, which have distinct real solutions when $\Delta > 0$. We can rearrange each equation and apply this principle to obtain two inequalities in θ:

$$\cos^2(\theta) > -4\sin(\theta)$$

$$\sin^2(\theta) > 4\cos(\theta)$$

We can use the Pythagorean identity to rewrite each equation in terms of one function of θ:

$$1 - \sin^2(\theta) > -4\sin(\theta) \Rightarrow 0 > \sin^2(\theta) - 4\sin(\theta) - 1$$

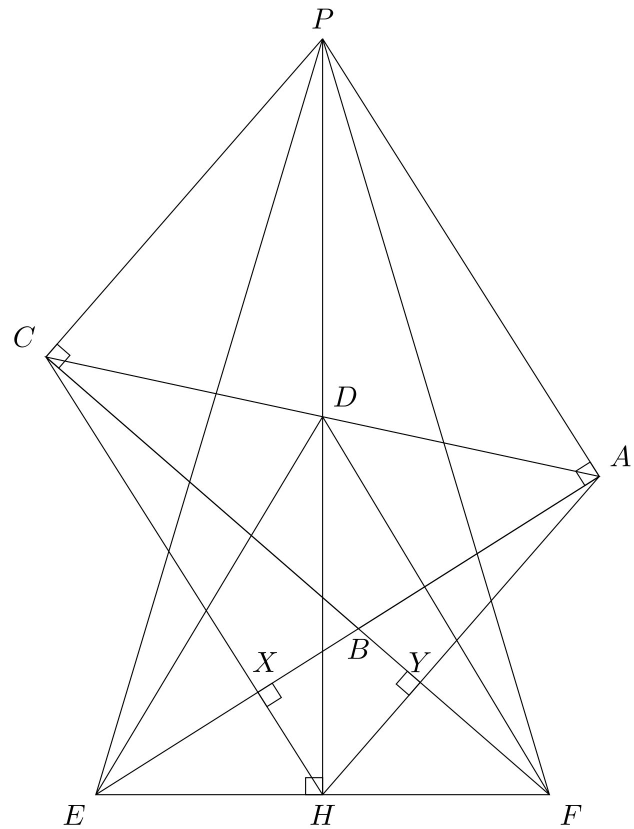
$$1 - \cos^2(\theta) > 4\cos(\theta) \Rightarrow 0 > \cos^2(\theta) + 4\cos(\theta) - 1$$

We can apply any method of solving quadratic equations to solve these quadratic inequalities for $\sin(\theta)$ and $\cos(\theta)$. As a result, we find that $\sin(\theta) > 2 - \sqrt{5}$ and $\cos\theta < \sqrt{5} - 2$. The unit circle is a useful tool for visualizing the interval of angles in $[0, 2\pi)$. Some visual reasoning should yield that the range of acceptable values of θ is $(\cos^{-1}(\sqrt{5}-2), \pi - \sin^{-1}(2-\sqrt{5}))$. Thus, the desired probability is $\frac{\pi+\sin^{-1}(\sqrt{5}-2)-\cos^{-1}(\sqrt{5}-2)}{2\pi}$ and the answer is $\boxed{\sqrt{5}-2}$.

16. Let H be the orthocenter of obtuse triangle $\triangle ABC$ and D be the midpoint of $\overline{AC}$. A line through H perpendicular to $\overleftrightarrow{DH}$ intersects with $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ at points E and F, respectively. Given that $EH = 3$ and $HD = 5$, compute the maximum possible area of triangle $\triangle DEF$.

Answer: 15

Solution:



We will show that regardless of the arrangement of $\triangle ABC$ (so long as it is obtuse), the area of $\triangle DEF$ will always equal $EH \cdot DH$.

Extend $\overline{HD}$ from D to P such that $HP = 2HD$, and connect segments $\overline{AP}$, $\overline{CP}$. Let the altitude from C intersect $\overleftrightarrow{AB}$ at X, and the altitude from A intersect $\overleftrightarrow{BC}$ at Y. Then, since the diagonals $\overline{AC}$ and $\overline{HP}$ bisect each other at D, $PAHC$ is a parallelogram. Thus, $\overleftrightarrow{CH} \parallel \overleftrightarrow{AP}$, $\overleftrightarrow{CP} \parallel \overleftrightarrow{AH}$. So we have $\angle PAE = \angle 90^\circ = \angle EHP$, and similarly on the other side $\angle PCF = \angle 90^\circ = \angle PHF$.

So E, H, A, P are concyclic (on a circle with diameter $\overline{EP}$), and H, F, P, C are also concyclic (on a circle with diameter $\overline{FP}$). Thus, by inscribed angle theorem we have that

$$\angle EPH = \angle EAH, \angle HPF = \angle HCF.$$

Then, since $\angle CXA = \angle AYC = 90^\circ$ and $\angle AHX = \angle CHY$ because A, Y, H and C, X, H are collinear (in that order, due to obtuseness), we have that $\triangle AHX \sim \triangle CHY$ and $\angle XAH = \angle YCH$.

Since A, E, X are collinear and C, F, Y are collinear we have that $\angle EAH = \angle HCF$, meaning that $\angle EPH = \angle FPH$: this makes the angle bisector of $\triangle EFP$ $\overline{PH}$, which is also its altitude. Thus, $\triangle EFP$ is isosceles and $EH = FH$. Finally, since $\overline{DH} \perp \overline{EF}$, we have the area of $\triangle DEF$ equal to $\frac{1}{2}DH \cdot (EH + FH) = DH \cdot EH$. Plugging in $DH = 5, EH = 3$ yields an area of $\boxed{15}$.

17. Let the *power*, $p(n)$, of a positive integer n be the number of fractions of the form $\frac{m}{n}$ that are in simplest form over all positive integers m with $1 \leq m \leq n$. ($\frac{1}{1}$ is in simplest form.) Let a positive integer n be *weak* if $\frac{p(n)}{n} \leq \frac{p(k)}{k}$ for all $1 \leq k < n$. Compute the sum of all *weak* positive integers less than 2024.

Answer: 10147

Solution: The first step is to realize that $p(n) = \varphi(n)$, Euler's totient function. Then, we use the fact that

$$\frac{\varphi(n)}{n} = \prod_{p|n} \frac{p-1}{p}$$

which comes from the multiplicativity of $\frac{\varphi(n)}{n}$ and the fact that $\varphi(p^k) = (p-1)p^{k-1}$ as p^k is coprime to every number that is not a multiple of p. So, in order to minimize this quantity, we want to maximize the number of distinct prime factors for small n. The sequence of weak numbers is therefore $1, 2 = 1(2), 2(2), 3(2) = 1(2\cdot 3), 2(2\cdot 3), 3(2\cdot 3), 4(2\cdot 3), 5(2\cdot 3) = (2\cdot 3\cdot 5), 2(2\cdot 3\cdot 5)\cdots$ where the pattern is that a weak number is a (small) multiple of the product of the first k prime numbers for some nonnegative integer k. The largest of these is $(2\cdot 3\cdot 5\cdot 7)\cdot 9 = 210\cdot 9$, since $210\cdot 10 = 2100 > 2024$.

To sum these integers, we break into 5 sums:

$$1 + \sum_{i=1}^{2} 2i + \sum_{i=1}^{4} 6i + \sum_{i=1}^{6} 30i + \sum_{i=1}^{9} 210i$$

Using the identity $\sum_{k=1}^{n} k = n(n+1)/2$, we obtain the final answer

$$1 + 2(3) + 6(10) + 30(21) + 210(45) = \boxed{10147}.$$

18. Arthur has a four-sided die, of which all faces are initially labeled 1. Every second, Arthur rolls the die and, if the outcome of the die is 1, then he changes the number on the top face to 2, and otherwise, he changes the number to 1. Let the probability that the sum of the die rolls is at some point k be p_k. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. Compute $\{p_1 + p_2 + \cdots + p_{2024}\}$.

Answer: $\mathbf{\frac{7}{8} - \frac{1}{2^{1351}}}$

Solution: Let us calculate p_i for a particular i. First, we actually consider the complement, call this q_i. In this event, we have that Arthur somehow gets to a sum of $i-1$, then rolls a 2.

We construct and solve a state based recurrence relation. Let $x_{i,j}$ be the probability that after i die rolls, there are j many 2s on Arthur's die. Importantly, we are able to recover the current sum of rolls from this information, which we will make use of later when calculating our answer.

Note that every roll flips the parity of the number of 2s on the die: in particular, this means that after an odd number of rolls there must be an odd number of 2s on the die and the same for evens. So, we have $x_{2i,1} = x_{2i,3} = x_{2i+1,0} = x_{2i+1,2} = x_{2i+1,4} = 0$. Since we know that $\sum_{j=0}^{4} x_{i,j} = 1$, we can conclude that $x_{2i,2} = \frac{3}{4}$ by the fact that $x_{2i-1,1} + x_{2i-1,3} = 1$ and both states are equally likely to move to a die with two 2s (turning a 1 to a 2 with probability 3/4 or a 2 to a 1 with probability 3/4). The next observation is that $x_{2i,0} - x_{2i,4} = \frac{1}{4^i}$. Conceptually, this is because after two rolls, we "lose" 3/4 of the difference to the "center" (with two 2s) and only 1/4 is reflected back to the edge (with zero or four 2s). This can be shown more rigorously by induction as well. Finally, we note that $x_{2i+1,1} - x_{2i+1,3} = x_{2i,0} - x_{2i,4}$, as a die with two 2s is equally likely to have one or three 2s after one roll, so that state does not affect the difference between those probabilities. Using the facts we have assembled, we can compute all of the probabilities by solving systems of linear equations; for $i > 0$, we have

- $x_{2i,1} = x_{2i,3} = x_{2i+1,0} = x_{2i+1,2} = x_{2i+1,4} = 0.$

- $x_{2i,2} = \frac{3}{4}$.
- $x_{2i,0} = \frac{1}{8} + \frac{1}{2 \cdot 4^i}$.
- $x_{2i,4} = \frac{1}{8} - \frac{1}{2 \cdot 4^i}$.
- $x_{2i+1,1} = \frac{1}{2} + \frac{1}{2 \cdot 4^i}$.
- $x_{2i+1,3} = \frac{1}{2} - \frac{1}{2 \cdot 4^i}$.

Now, we compute the probability of rolling a certain sum. Note that if after i rolls we have j 2s on the die, this means we have rolled j more 1s than we have 2s. Therefore, we've rolled $\frac{i+j}{2}$ 1s and $\frac{i-j}{2}$ 2s, giving a total sum of $\frac{3i-j}{2}$. Importantly, the difference between any two sums taken from the same number of rolls is at most 2 (since $0 \le j \le 4$), and each increase of i by 2 increases the sum by 3 (if j remains the same). So, we consider sums mod 3 (specifically, $3i-1, 3i, 3i+1$) and compute the probability of rolling a 2 in those situations by casework:

$$\begin{aligned} q_{3i} &= \frac{1}{2} x_{2i,2} \\ q_{3i+1} &= 0 \cdot x_{2i,0} + \frac{3}{4} x_{2i+1,3} \\ q_{3i+2} &= \frac{1}{4} x_{2i+1,1} + 1 \cdot x_{2i+2,4} \end{aligned}$$

Substituting our expressions and simplifying gives us $q_{3i} = \frac{3}{8}, q_{3i+1} = \frac{3}{4} \cdot \left(\frac{1}{2} - \frac{1}{2^{2i+1}}\right) = \frac{3}{8} - \frac{3}{4} \cdot \frac{1}{2^{2i+1}}, q_{3i+2} = \frac{1}{4}$. Then, we compute

$$\begin{aligned} \sum_{i=1}^{2024} q_i &= \left(\sum_{i=0}^{674} q_{3i+1}\right) + \left(\sum_{i=0}^{674} q_{3i+2}\right) + \left(\sum_{i=0}^{673} q_{3i+3}\right) \\ &= \left(\sum_{i=0}^{674} \frac{3}{8} + \frac{3}{4} \cdot \frac{1}{2^{2i+1}}\right) - \left(\sum_{i=0}^{674} \frac{1}{4}\right) + \left(\sum_{i=0}^{673} \frac{3}{8}\right) \\ &= -\frac{3}{8} + \sum_{i=0}^{674}\left(\frac{3}{8} + \frac{1}{4} + \frac{3}{8}\right) - \frac{3}{4}\left(\sum_{i=0}^{674} \frac{1}{2^{2i+1}}\right) \\ &= -\frac{3}{8} + 675 - \frac{3}{4}\left(\frac{2}{3} - \sum_{i=675}^{\infty} \frac{1}{2^{2i+1}}\right) \\ &= -\frac{3}{8} + 675 - \frac{3}{4}\left(\frac{2}{3} - \frac{2}{3} \cdot \frac{1}{2^{2 \cdot 675}}\right) \\ &= -\frac{3}{8} + 675 - \left(\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^{1350}}\right) \\ &= -\frac{3}{8} + 675 - \left(\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^{1350}}\right) \\ &= 675 - \frac{7}{8} + \frac{1}{2^{1351}}. \end{aligned}$$

Now, we have that $\sum_{i=1}^{2024} p_i = 2024 - \sum_{i=1}^{2024} q_i = (2024 - 675) + \frac{7}{8} - \frac{1}{2^{1351}}$. Therefore, the answer is $\boxed{\frac{7}{8} - \frac{1}{2^{1351}}}$.

19. Let N_{21} be the answer to problem 21. Let b and d be real numbers such that the polynomial $P(x) = x^4 - N_{21}x^3 + bx^2 - x + d$ has real roots p, q, r, s and $pq = rs$. Compute the greatest possible integer value of $P(N_{21})$.

Answer: 15

Solution: Using Vieta's relations, we have the following equations relating p, q, r, and s to the coefficients of the polynomial:

$$p + q + r + s = N_{21}$$

$$pq + pr + ps + qr + qs + rs = b$$

$$pqr + pqs + prs + qrs = 1$$

$$pqrs = d$$

In particular, we can do some nice manipulations of the second and fourth equations using the fact that $pq = rs$:

$$pq + pr + ps + qr + qs + (pq) = p(q + r + s) + q(p + r + s) = p(N_{21} - p) + q(N_{21} - q) = b$$

$$pqr + pqs + p(pq) + q(pq) = pq(p + q + r + s) = N_{21}pq = 1 \implies pq = \frac{1}{N_{21}}$$

Since $pq = rs = \frac{1}{N_{21}}$ and $d = pqrs$, we have $d = \frac{1}{N_{21}^2}$. At this point, we have determined exact values for almost all the coefficients of our polynomial. The x^4, x^3, x, and constant coefficients are all determined. Thus, to maximize $P(N_{21})$ it suffices to maximize b because $x^2 \geq 0$ for all real x (including N_{21}, which is of particular interest to us!).

We have reduced the problem to maximizing $b = p(N_{21} - p) + q(N_{21-q})$ subject to the constraint that $pq = \frac{1}{N_{21}}$. Expanding our expression for b, we are equivalently maximizing $-p^2 - q^2 + (p + q)(N_{21})$, or minimizing $p^2 + q^2 - (p + q)(N_{21})$. This expression is suspiciously similar to the expansion of $(p + q - \frac{N_{21}}{2})^2 = p^2 + q^2 - (p + q)(N_{21}) + 2pq + \frac{N_{21}^2}{4}$; in fact, it is so similar that it only differs by $2pq + \frac{N_{21}^2}{4}$ which is a constant (within the confines of this problem)!

Clearly then, the values of p and q which maximize b will also minimize $(p + q - \frac{N_{21}}{2})^2$. This is a square of a real number since p and q are real, so it takes its minimum value of 0 when $p + q - \frac{N_{21}}{2} = 0$. The expression for $-b$ is less than $(p + q - \frac{N_{21}}{2})^2$ by $2pq + \frac{N_{21}^2}{4}$, so the maximum value of b is $2pq + \frac{N_{21}^2}{4} = \frac{2}{N_{21}} + \frac{N_{21}^2}{4}$.

We have determined that the polynomial which maximizes $P(N_{21})$ is

$$x^4 - N_{21}x^3 + \left(\frac{2}{N_{21}} + \frac{N_{21}^2}{4}\right)x^2 - x + \frac{1}{N_{21}^2}$$

and so the maximum possible value of $P(N_{21})$ under the given constraints is $\frac{1}{4}N_{21}^4 + N_{21} + \frac{1}{N_{21}^2}$.

We have to round down to the nearest integer since the problem requests the greatest possible integer value of $P(N_{21})$. Thus, the answer to this problem is $\lfloor \frac{1}{4}N_{21}^4 + N_{21} + \frac{1}{N_{21}^2} \rfloor$.

Importantly, we are guaranteed that this value is actually attainable since b can be any real number less than or equal to $\frac{2}{N_{21}} + \frac{N_{21}^2}{4}$.

Plugging in $N_{21} = \frac{560}{209}$ gives the answer $\boxed{15}$.

20. Let N_{19} be the answer to problem 19. Danielle picks a real number p uniformly at random from $[0, 1]$. She then creates a magic coin that has probability p of landing on heads and probability $1-p$ of landing on tails when flipped. Compute the probability that Danielle lands heads exactly N_{19} times in $5N_{19}$ flips of the coin.

Answer: $\frac{1}{76}$

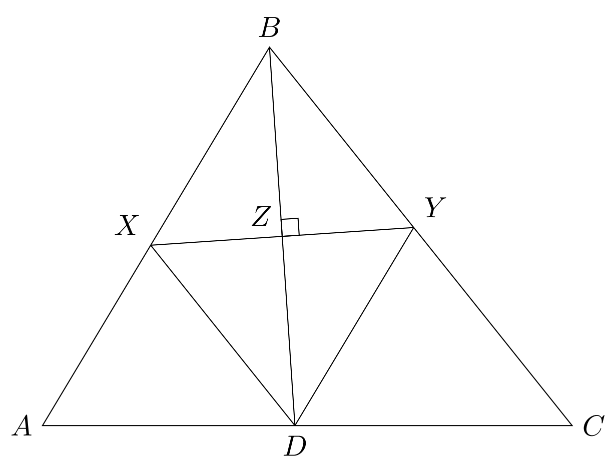
Solution: Let C be a random variable corresponding to selecting a value uniformly at random from $[0, 1]$. The value of C represents the chosen probability p of flipping a heads on Danielle's magic coin. Then, let $X_1, X_2, \cdots, X_{5N_{19}}$ be independent random variables also corresponding to selecting a value uniformly at random from $[0, 1]$. Note that the probability $\mathbb{P}(X_i < C | C = p) = p$ for any given X_i and fixed value p of C. So, we can consider the number of heads flipped to be the number of random variables $X_{i_1}, X_{i_2}, \cdots, X_{i_m}$ that are less than p. We can then consider the probability of flipping exactly N_{19} heads as the probability that the list $C, X_1, X_2, \cdots, X_{5N_{19}}$ when sorted in ascending order has C as the $N_{19}+1$st element in the list. However, since C and all of the X_i are independent, identically distributed uniform random variables, by symmetry it is equally likely for C to be the kth smallest element for any $1 \leq k \leq 5N_{19}+1$ in this list. Therefore, the probability that $k = N_{19}$ is exactly $\frac{1}{5N_{19}+1}$.

Plugging in $N_{19} = 15$ yields the answer $\boxed{\frac{1}{76}}$.

21. Let N_{20} be the answer to problem 20. In triangle $\triangle ABC$, the angle bisector of $\angle B$ intersects $\overline{AC}$ at D. The perpendicular bisector of $\overline{BD}$ intersects $\overline{AB}$ and $\overline{BC}$ at X and Y respectively. The area of $\triangle BXY$ is $\frac{7}{55}$, $AD = \frac{1}{4}$, and $DC = N_{20}$. Compute the area of $\triangle ABC$.

Answer: $\frac{560}{209}$

Solution:



[Diagram not to scale.]

Let BD and XY intersect at Z. Since $\angle YBZ = \angle XBZ$ and $\angle XZB = \angle YZB = 90°$, we have $\triangle XZB \sim \triangle YZB$. Additionally, since $ZB = ZB$, we have $\triangle XZB \cong \triangle YZB$.

Now connect X to D and Y to D and consider the triangles $\triangle XZD$ and $\triangle YZD$. Due to the perpendicular bisector, we have $ZD = ZB$ and $\angle XZD = \angle YZD = 90°$. We find that there aren't just two congruent triangles in this configuration, but actually four! In particular, $\triangle XZB \cong \triangle YZB \cong \triangle XZD \cong \triangle YZD$.

As a result, we have $BX = XD = DY = YB$, so $BXDY$ is a rhombus and therefore a parallelogram. Thus, $XD||BC$ and $DY||AB$, so $\triangle AXD \sim \triangle ABC \sim \triangle DYC$.

Using similarity, we have

$$\frac{AX}{AB} = \frac{AD}{DC} = \frac{\frac{1}{4}}{\frac{1}{4} + N_{20}}$$
$$\frac{CY}{CB} = \frac{DC}{AC} = \frac{N_{20}}{\frac{1}{4} + N_{20}}$$

The area of $\triangle ABC$ is given by $\frac{1}{2}\sin\angle B \cdot BA \cdot BC$, which we can rewrite as $\frac{1}{2}\sin\angle B \cdot BX \cdot BY \cdot \frac{AB}{BX} \cdot \frac{BC}{BY}$ This is convenient because we have $\frac{1}{2}\sin\angle B \cdot BX \cdot BY = [\triangle BXY]$, which is known to us! Thus, the desired area is equivalent to $[\triangle BXY] \cdot \frac{AB}{BX} \cdot \frac{BC}{BY} = \frac{7}{55} \cdot \frac{\frac{1}{4}+N_{20}}{\frac{1}{4}} \cdot \frac{\frac{1}{4}+N_{20}}{N_{20}}$ and we're done! The desired area is $\frac{7}{55} \cdot \frac{(\frac{1}{4}+N_{20})^2}{\frac{1}{4}N_{20}}$.

Plugging in $N_{20} = \frac{1}{76}$ gives the final answer, $\boxed{\frac{560}{209}}$.

By solving problems 19, 20, and 21 we find the following relationships between N_{19}, N_{20}, and N_{21}:

$$N_{19} = \left\lfloor \frac{N_{21}^4}{4} + N_{21} + \frac{1}{N_{21}^2} \right\rfloor$$
$$N_{20} = \frac{1}{5N_{19} + 1}$$
$$N_{21} = \frac{7}{55} \cdot \frac{(\frac{1}{4} + N_{20})^2}{\frac{1}{4}N_{20}}$$

Based on these relationships, we can make some crude observations and predictions about the exact values of N_{19}, N_{20}, and N_{21}. First, it is apparent that all the answers are nonnegative. In addition, since N_{19} is a nonnegative integer, N_{20} is likely to be quite small. Even modest values of N_{19} cause N_{20} to be much less than 1; for example, in the case $N_{19} = 3$, we'd have $N_{20} = \frac{1}{16}$.

Looking at the last equation, the observation that N_{20} is small becomes useful. We see that we can expand as

$$N_{21} = \frac{7}{55} \cdot \frac{\frac{1}{16} + \frac{1}{4}N_{20} + N_{20}^2}{\frac{1}{2}N_{20}} \approx \frac{7}{55} \cdot \frac{\frac{1}{16} + \frac{1}{2}N_{20}}{\frac{1}{4}N_{20}}$$

where the N_{20}^2 term is likely to be so small compared to $\frac{1}{16}$ as to be insignificant. The cases where N_{20}^2 is closer in magnitude to $\frac{1}{16}$ such that this approximation introduces significant error are those where N_{19} is small, really only $N_{19} = 0, 1, 2$, which can be easily checked and disregarded.

Thus, we have rewritten the last equation in a much more tractable form:

$$N_{21} \approx \frac{7 + 56N_{20}}{220N_{20}}$$

Rearranging this and solving for N_{20} gives

$$N_{20} \approx \frac{7}{220N_{21} - 56}$$

which we can use to solve for N_{19} in terms of N_{21}:

$$\frac{7}{220N_{21} - 56} \approx N_{20} = \frac{1}{5N_{19} + 1} \implies N_{19} \approx \frac{220N_{21}}{35} - \frac{9}{5}$$

This can be used with the first equation to write an equation only in terms of N_{21}. Note the choice to solve in terms of this variable; if we had chosen another one, then we would have had to deal with substitutions for N_{21} in $\frac{N_{21}^4}{4} + N_{21} + \frac{1}{N_{21}^2}$ which would be very painful :) We have the following:

$$\frac{44N_{21}}{7} - \frac{9}{5} \approx \left\lfloor \frac{N_{21}^4}{4} + N_{21} + \frac{1}{N_{21}^2} \right\rfloor = N_{19}$$

This looks very scary; it's a quartic equation with extra complexity due to the floor function. However, with a few tricks we can reasonably approximate the value of N_{21} such that this equation holds. For starters, we'll mostly do analysis of the magnitude of each side of the equation to get reasonable bounds on the solutions, so we'll ignore the floors.

Since $\frac{N_{21}^4}{4}$ grows very quickly with N_{21} we can immediately bound any solution above by 3; at $N_{21} = 3$ the RHS of this equation is about 23, and the LHS is just above 17. Beyond this point, the RHS is much much larger than the LHS. Additionally, plugging in $N_{21} = 2$ causes the LHS to be greater than the RHS, so we know a solution to the equation exists for $2 < N_{21} < 3$. To get even more granular, we can try $N_{21} = 2.5$ and observe that the LHS is still greater than the RHS (LHS is roughly 14, RHS is roughly 12), so a solution $2.5 < N_{21} < 3$ must exist. As we may realize, however, the most useful outcome of this analysis is not even the bound on N_{21} but instead the bound on N_{19}. At our solution point we will have $N_{19} \approx \text{RHS} = \text{LHS}$, which is greater than 14 and less than 18.

N_{19} is an integer, so it equals 14, 15, or 16. These give $N_{20} = \frac{1}{71}, \frac{1}{76}, \frac{1}{81}$, respectively. Using our approximation for N_{21} to minimize intermediate computations (we'll have to use the exact formula to extract the answer to problem 21, but we don't really want to do this work more than once!), we get $N_{21} = \frac{553}{220}, \frac{588}{220}, \frac{623}{220}$ respectively. $N_{21} = \frac{553}{220}$ is certainly too small by our previous work since it's extremely close to 2.5. $\frac{623}{220}$ is too big; a good approximation for this number is $\frac{17}{6}$ and plugging this into the first equation gives $N_{19} \neq 16$, a contradiction. Thus, we must have $N_{21} \approx \frac{588}{220} \approx \frac{8}{3}$. We can easily verify that this works and therefore $N_{19} = 15$.

We do have to be a little careful; we assumed the desired solution to

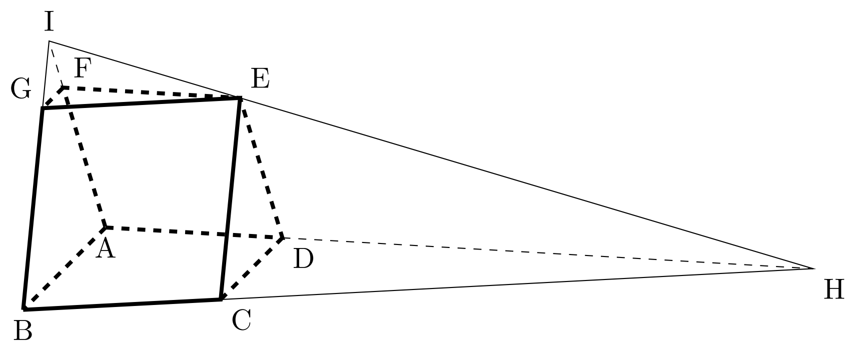
$$\frac{44N_{21}}{7} - \frac{9}{5} \approx \left\lfloor \frac{N_{21}^4}{4} + N_{21} + \frac{1}{N_{21}^2} \right\rfloor = N_{19}$$

was near $N_{21} = 2.5$. However, seeing the $\frac{1}{N_{21}^2}$ term may make us (appropriately) concerned that there is actually another solution where N_{21} is between 0 and 1. With a quick exploration, we can bound this solution between $\frac{2}{3}$ and 1 and see that this case implies $N_{19} = 2$ which is a case we've already verified doesn't work.

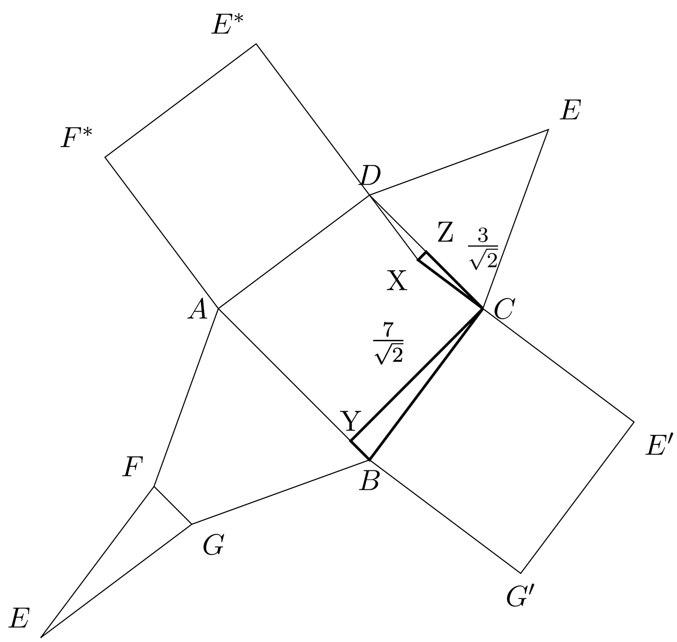
22. Tetrahedron $ABIH$ has points C on $\overline{BH}$, D on $\overline{AH}$, E on $\overline{HI}$, F on $\overline{AI}$, and G on $\overline{BI}$ such that the squares $BCEG$ and $ADEF$ have side length 5. Quadrilaterals $ABCD$ and $ABGF$ are isosceles trapezoids. Given that $AB = 4\sqrt{2}$ and $CD = 3\sqrt{2}$, compute the volume of solid $ABCDEFG$.

Answer: $\mathbf{20\sqrt{10}}$

Solution: Notice that $FGEI$ and $DCHE$ are similar tetrahedron to $ABHI$. Thus $\frac{CD}{AB} = \frac{3}{4} = \frac{CH}{BH}$, so the side lengths of $DCHE$ are $\frac{3}{4}$ the side lengths of $ABHI$. Since $BCEG$ is a square, we get that $\frac{CE}{CH} = \frac{BC}{CH} = \frac{1}{3}$. Since $\triangle IGE \sim \triangle ECH$, we get $\frac{1}{3} = \frac{GI}{EG} = \frac{GI}{BG}$. This tells us that the side lengths of $FGEI$ are $\frac{1}{4}$ the side lengths of $ABHI$.



We want to project point E down to plane $ABCD$ to solve for the height of $ABHI$. Let's use the net for $ABCDEFG$ to figure out where the projection of E lies.



When rotating the square ADE^*F^* around $\overline{AD}$, notice that the projection of E^* always lands on $\overleftrightarrow{DE^*}$. Similarly, the projection of E' down to the plane $ABCD$ must lie on $\overleftrightarrow{CE'}$. Thus, the projection of E onto the plane $ABCD$ must lie on the intersection of $\overleftrightarrow{DE^*}$ and $\overleftrightarrow{CE'}$. So, we extend $\overline{CE'}$ and $\overline{DE^*}$ to point X, which is the projection of E down to plane $ABCD$.

To find the height EX, we draw altitudes $\overline{XZ}$ and $\overline{CY}$. See that $BY = \frac{AB-CD}{2} = \frac{1}{\sqrt{2}}$ so $CY = \sqrt{BC^2 - BY^2} = \frac{7}{\sqrt{2}}$.
Now, see that $\angle XCZ \cong \angle BCY$ since both have measure $\angle BCD - 90°$. This means $\triangle BCY \sim \triangle XCZ$. Thus, $XZ = \frac{3}{7\sqrt{2}}$. Use the right triangle $\triangle CZE$ to get $EZ^2 = CE^2 - CZ^2 = \frac{41}{2}$. Now, if we look in the solid $ABCDEFG$ we can use right triangle $\triangle EXZ$ to get $EX = \sqrt{EZ^2 - XZ^2} = \frac{10\sqrt{10}}{7}$. To find the volume of $DCHE$, we see that the area of $\triangle CDH$ is $\frac{1}{2}CD \cdot HZ = \frac{1}{2}(3\sqrt{2})(3 \cdot \frac{7}{\sqrt{2}}) = \frac{63}{2}$. So, the volume of $DCHE$ is $\frac{1}{3} \cdot \frac{63}{2} \cdot \frac{10\sqrt{10}}{7} = 15\sqrt{10}$ This is $(\frac{3}{4})^3 = \frac{27}{64}$ the volume of $ABHI$ but we want $1 - (\frac{1}{4})^3 - (\frac{3}{4})^3 = \frac{36}{64}$. Thus, our desired volume is $\frac{36}{27} \cdot 15\sqrt{10} = 20\sqrt{10} = \sqrt{4000} = \boxed{20\sqrt{10}}$.

23. Let X denote the set $\{-1, 0, 1, 2, 3, 4\}$, and let $\mathcal{P}(X)$ denote the set of all subsets of X. Compute the number of functions $f\colon \mathcal{P}(X) \to X$ such that $f(\varnothing) = 0$ and $f(A \cap B) + f(A \cup B) = f(A) + f(B)$ for any subsets A and B of X.

Answer: 966

Solution: For brevity, set $n = 4$. Call a function f *special* if and only if $f(\varnothing) = 0$ and $f(A \cap B) + f(A \cup B) = f(A) + f(B)$. We classify all special functions. On one hand, let f be a special function. Then for any subset $A \subseteq X$ and element $x \in X \setminus A$, we see that

$$f(A \cup \{x\}) = f(\varnothing) + f(A \cup \{x\}) = f(\{x\}) + f(A)$$

because f is special. Rearranging, we see that $f(A \cup \{x\}) = f(A) + f(\{x\})$. Thus, an induction on the number of elements of A implies that

$$f(A) = \sum_{x \in A} f(\{x\}).$$

And conversely, for any f defined as above, we can check that

$$\begin{aligned} f(A \cap B) + f(A \cup B) &= \sum_{x \in A \cup B} f(\{x\}) + \sum_{x \in A \cap B} f(\{x\}) \\ &= \sum_{x \in A} f(\{x\}) + \sum_{x \in B} f(\{x\}) \\ &= f(A) + f(B). \end{aligned}$$

The point is that f is completely determined by its values on singletons.

To compute the desired number of functions f, it remains to deal with the condition that $-1 \le f(A) \le n$ for any subset $A \subseteq X$. Notably, $0 \le f(\{x\}) \le n$. But the chief difficulty lies in the fact that $f(\{x\})$ may be nonpositive. For example, note

$$f(A) = \sum_{x \in A} f(\{x\})$$

is at least -1 for all subsets $A \subseteq X$ if and only if $f(\{x\}) = -1$ for at most one value of x. So we might as well do casework.

- Suppose $f(\{-1\}) = -1$. Then

$$f(A) = \sum_{x \in A} f(\{x\}) \le \sum_{x=0}^{n} f(\{x\}) = f(X \setminus \{-1\}),$$

so it remains to check $\sum_{x=0}^{n} f(\{x\}) \leq n$. From here, the number of possible f is the the number of ways to choose $n+1$ nonnegative integers to have sum at most n. This is equal to the number of ways to choose $n+2$ nonnegative integers with sum exactly n, which is $\binom{2n+1}{n}$.

- Suppose $f(\{x\}) \geq 0$ for each x. Arguing as above, it remains to check $\sum_{x=-1}^{n} f(\{x\}) \leq n$, so we are counting the number of ways to choose $n+2$ nonnegative integers to have sum at most n. Computing as above, this is $\binom{2n+2}{n}$.

Combining the above work, we total to

$$(n+2)\binom{2n+1}{n} + \binom{2n+2}{n},$$

which is $6\binom{9}{4} + \binom{10}{4} = \boxed{966}$.

24. A polynomial with integer coefficients that has a root of the form $k\cos\left(\frac{4\pi}{7}\right)$ for some positive integer k is called *simple* if there are no polynomials of lesser degree with integer coefficents sharing the same root. There exists a unique simple polynomial $P(x)$ with leading coefficient 1 such that $|P(3)|$ is minimized over all simple polynomials with leading coefficient 1. Compute $P(4)$.

Answer: 56

Solution: First, let $z = e^{\frac{4\pi i}{7}}$. This is a root of $z^7 - 1 = (z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$. Since $z \neq 1$, we must have $z^6+z^5+z^4+z^3+z^2+z+1 = 0$. This is the minimal polynomial of z over the rational numbers, and also notice that it is a cyclotomic polynomial. To achieve minimal degree, we can divide this by z^3. This gives:

$$\frac{z^6+z^5+z^4+z^3+z^2+z+1}{z^3} = z^3+z^2+z+1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3} = z^3+\frac{1}{z^3}+z^2+\frac{1}{z^2}+z+\frac{1}{z}+1.$$

We then have

$$z+\frac{1}{z} = e^{\frac{4\pi i}{7}} + e^{-\frac{4\pi i}{7}} = 2\cos\left(\frac{4\pi}{7}\right).$$

Let $x = z + \frac{1}{z}$. Notice that

$$z^3 + \frac{1}{z^3} = \left(z+\frac{1}{z}\right)^3 - 3\left(z+\frac{1}{z}\right) = x^3 - 3x.$$

Similarly,

$$z^2 + \frac{1}{z^2} = \left(z+\frac{1}{z}\right)^2 - 2 = x^2 - 2.$$

Therefore, $z^3 + \frac{1}{z^3} + z^2 + \frac{1}{z^2} + z + \frac{1}{z} + 1 = (x^3-3x) + (x^2-2) + x + 1 = x^3 + x^2 - 2x - 1 = 0$. In this case, we have shown there exists a minimal polynomial with $k = 2$ and degree 3.

Now, we claim $P_1(x_1) = x_1^3 + x_1^2 - 2x_1 - 1$ is unique, and we will show this by contradiction. Assume there exists $Q(x_1)$ with degree 3 such that $k\cos\left(\frac{4\pi}{7}\right)$ is a root of Q. Then the difference between a monic polynomial with degree 3 and rational coefficients $R(x_1)$ and $Q(x_1)$ will be $R(x_1) - Q(x_1) = S(x_1)$ for some polynomial $S(x_1)$ with degree less than 3 and non-integer coefficients. We have $S(k\cos(\frac{4\pi}{7})) = 0$, which means $S(x_1)$ has degree less than $P(x_1)$, which is a contradiction to the fact that $P_1(x_1)$ has minimal degree. Thus, $P_1(x_1)$ is unique.

However, we need to ensure that $|P(3)|$ is minimized. We can plug in $\frac{2x}{k}$ for x in the polynomial $P_1(x_1)$ because it give us a root of $k\cos(\frac{4\pi}{7})$ as desired.

This gives

$$P_1\left(\frac{2x}{k}\right) = \left(\frac{2x}{k}\right)^3 + \left(\frac{2x}{k}\right)^2 - 2\left(\frac{2x}{k}\right) - 1 = x^3 + \frac{kx^2}{2} - \frac{k^2x}{2} - \frac{k^3}{8}$$

after transforming it back to a monic polynomial.

Since $P_1(x_1)$ must have integer coefficients, clearly k is even. Therefore, if $k = 2n$ for some integer n we have $P_1(x_1) = x_1^3 + nx_1^2 - 2n^2x_1 - n^3$, and more specifically, $|P(3)| = |27+9n-6n^2-n^3|$. We look to find the minimum of $|27+9n-6n^2-n^3|$ for $n \in \mathbb{Z}^+$. We know the cubic $27+9n-6n^2-n^3$ goes to infinity on the left and negative infinity on the right, and a cubic can only change direction twice. We see that

$$\begin{aligned}
27 + 9(0) - 6(0)^2 - (0)^3 &= 27 \\
27 + 9(1) - 6(1)^2 - (1)^3 &= 29 \\
27 + 9(2) - 6(2)^2 - (2)^3 &= -13 \\
27 + 9(3) - 6(3)^2 - (3)^3 &= -27
\end{aligned}$$

where the function decreases at some point before $n = 0$, increases after $n = 0$, and decreases again after $n = 1$, meaning that it has already changed direction twice, and it must therefore continue to decrease after $n = 2$. Thus, we conclude the absolute value of the cubic is minimized on the positive integers at $n = 2$ for a value of 13. Therefore, we can plug in $n = 2$ to $p(x_1)$: $p(x_1) = x_1^3 + 2x_1^2 - 8x_1 - 8$.

The desired polynomial is $P(x) = x^3 + 2x^2 - 8x - 8$, with $P(4) = 4^3 + 2(4)^2 - 8(4) - 8 = \boxed{56}$.

25. The Fibonacci numbers F_n for integers $n \geq 1$ are defined as follows: $F_1 = F_2 = 1$, and for $n > 2$, $F_n = F_{n-1} + F_{n-2}$. Kiran makes a list of all the distinct positive integers less than or equal to 10^6 that can be expressed as the sum of at most four distinct Fibonacci numbers. Compute the length of Kiran's list. Submit your answer as an integer E; if the correct answer is A, your score for this question will be $\max\left(0, 25 - \left\lfloor \frac{\sqrt{|A-E|}}{4} \right\rfloor\right)$.

Answer: 17692

Solution: First, we want to count the number of Fibonacci numbers under 10^6. There's a few ways we could do this. One way is to compute forward to $F_{16} = 977$, and since the Fibonacci sequence is roughly geometric, we expect that $\frac{F_{16}}{F_1} \approx \frac{F_{31}}{F_{16}}$ and so F_{31} is roughly 10^6. Another strategy is to compute all the way to F_{31} by hand, or to treat F_{16} as 1000 and then compute forward with some loss of precision to speed up calculations. In general, we get that F_{30} is the largest Fibonacci number less than 10^6 (with a value roughly around $8 \cdot 10^5$ if we compute more precisely).

A naive method is to guess that we can choose any four of the first 30 Fibonacci numbers at random and sum them to get a distinct integer, which would give a guess of $\binom{30}{4} = 27405$, which will get only 1 point. However, we should also note that $F_1 = F_2$, so any instance where we choose F_1 and not F_2 is the same as choosing F_2 and not F_1. Therefore, we can remove about a quarter of the possibilities (There are four subsets of $\{0, 1\}$, and we're removing one for double counting). This gives $\frac{3}{4} \cdot \binom{30}{4} = 20653.75$, which (when rounded to an integer) scores 12 points. Another improvement is realizing that F_{30} is close to 10^6, so we should not choose it to guarantee that our sum is below 10^6. This gives $\frac{3}{4} \cdot \binom{29}{4} = 17813.25$, which (when rounded) earns 23 points. We are getting somewhat "lucky" here, since we made a lot of simplifying assumptions that happen to roughly cancel out to within 150 of the correct answer.

Another approach is to realize that adjacent Fibonacci numbers in the sequence can be combined into one Fibonacci number by the recurrence relation, which means to optimize for distinct sums we should pick nonadjacent Fibonacci numbers. In general, we expect a choice of some F_k to account for 3 of our choices: F_k and its two neighbors. We still leave out F_{30} as generally too large, but we no longer need to account for $F_1 = F_2$ since they are adjacent. This leads us to the estimation $29 \cdot 26 \cdot 23 \cdot 20/4! = 14451.666\ldots$, which earns 11 points. However, now that we've spaced the Fibonacci numbers apart, we should account for the cases where we select less than 4 distinct numbers. This gives a better estimation $29 \cdot 26 \cdot 23 \cdot 20/4! + 29 \cdot 26 \cdot 23/3! + 29 \cdot 26/2 + 29 = 17748$ earning 24 points. Only summing the first two terms and ignoring the other two as tiny gives 17342 scoring 21 points as well.

26. Submit a positive real number c to at most 6 decimal places. Define the function $f^1(x) = x^2 + c$, and let $f^k(0) = f^1(f^{k-1}(0))$ for $k \geq 2$. Let N be the smallest positive integer such that $f^N(0) > 2024$. If such an N does not exist, your score is 0 points. Otherwise, your score is $\max(0, 25 - 3|N - 20|)$ points.

Answer: $\mathbf{[0.277648, 0.280858]}$

Solution: The given answer is the range of values that scores 25 points. Range of submissions that score points: $[0.263591, 0.354188]$.

Let $f(x) = f^1(x)$ for convenience. The first thing to observe is that if we choose $c = 0.25$, we'll have $f(x) - x = x^2 + 0.25 - x = (x - 0.5)^2$. Since the distance from x to 0.5 is $(0.5 - x)$ and $(0.5 - x) > (0.5 - x)^2$ whenever $0 < x < 1$, we can see that $f^m(x) = f^{m-1}(x) + (0.5 - f^{m-1}(x))^2$ will never exceed 0.5 by induction. So, our lower bound for c is 0.25.

Then, roughly independently of c, once $f^m(c) \approx 2$, it will take about 4 more iterations to reach 2024: that is, $f^{m+4}(c) > 2024$ if $f^m(c) \approx 2$. Assuming $0.25 < c < 2$, we have

$$2, 4 < 4 + c < 6, 16 < 16 + 2c + c^2 < 36, 256 < 256 + \cdots < 1296, 2024 \ll 256^2 < 256^2 + \cdots < 1296^2$$

From here, the method we use to estimate an appropriate value of c is very ad-hoc, and it could be easier to simply compute (with limited precision) values for some random values of c and continue refining your guess that way. So, we want to reach 2 in 16 steps. Let $d = c - 0.25$, and we assume d is fairly small. Then $f(x) - x = (0.5 - x)^2 + d$, where $(0.5 - x)^2$ is fairly small after the first step (less than c^2). This means that getting from the first step c to $0.5 + c$ takes a little less than $\frac{0.75 - c}{d} = \frac{0.5 - d}{d}$ steps.

Now, we have $f(0.5 + c) = f(0.75 + d) = (0.75 + d)^2 + c = 0.5625 + 1.5d + d^2 + c = 0.8125 + 2.5d + d^2$ is close to 1. It's better to overestimate how far we get since we're ignoring the effects of small values, so we round up to 1, and suppose that 1 leads to $1 + c$ leads to a little less than 2, which will then take us to > 2024 in 4 steps. Therefore, we guess it takes roughly 7 steps to get from $0.5 + c$ to > 2024, and roughly $\frac{0.5 - d}{d} + 1$ steps to get from 0 to $0.5 + c$. This means we want to solve $\frac{0.5 - d}{d} + 1 = 13$, which gives us $13d = 0.5$ or $d \approx 0.038$. Adding 0.25 to get c gives an estimate of 0.288, which takes 18 steps to exceed 2024 and gets 19 points.

At this point, the easiest way to improve your estimate is likely to guess slightly lower and perform the computation manually to see what you could get: we'll want to fudge downwards wherever we can to make up for ignoring a lot of d^2 terms. Another strategy is to compute the true number of steps for 0.288 manually, and then add the difference to the number of steps we want out of $\frac{0.5-d}{d}+1$. This tells us to set $\frac{0.5-d}{d}+1=15$, giving $15d=0.5$ or $d=0.0333$. That gives an estimate of 0.283333, which gets you to 19 steps and 22 points. Iterating on this will give you a full scoring answer after 2 more attempts, giving a value of $c=0.25+1/34\approx 0.279412$. You could also attempt to get a better estimate of the effect of the $(0.5-x)^2$ term in $f(x)-x$, or manually compute forward until the distance between x and 0.5 is small enough that $(0.5-x)^2$ is negligible to handle the error generated on the left side.

A guess of $c=0.28$ earns full points and is likely the easiest full-scoring value to guess from this point.

27. Oliver rolls a standard, fair 6-sided die 640 times. Oliver tells Tushar that no 5s were rolled before he rolled a 6 (it is possible he rolled no 5s or 6s). Given this information, Tushar computes x, the expected number of 6s rolled by Oliver. Compute $\log_5(\lceil 6x\rceil - 6x)$. Submit your answer as a real number E to at most 3 decimal places; if the correct answer is A, you will receive $\max(0, 25-\lfloor 4|A-E|\rfloor)$ points.

Answer: -157.21443351782

Solution: Let $N=640$ for convenience. Consider some sequence of N rolls S, and let $f(S)$ be the sequence of rolls obtained by swapping all 5s and 6s. If S has a 5 or a 6, then exactly one of $S, f(S)$ follows the rule that Oliver did not roll a 5 before a 6. Therefore, the total number of 5s and 6s in all possible valid sequences S is equal to half the total number of 5s and 6s in all sequences of N rolls, regardless of validity (since f does not change the total number of 5s and 6s in a sequence).

The expected number of 5s and 6s in a valid sequence is then the total number of 5s and 6s in all valid sequences divided by the number of valid sequences. The first quantity is half the total number of 5s and 6s in all sequences given by our earlier argument, which is $\frac{1}{2}\cdot 2N6^{N-1}=N6^{N-1}$: there are $2\cdot 6^{N-1}$ sequences with a 5 or 6 in any given position, and multiplying by N counts the number of 5s or 6s in all positions. The total number of valid sequences is $\frac{6^N+4^N}{2}$ by the fact that there are 6^N-4^N sequences with a 5 or a 6, of which exactly half are valid, and 4^N sequences with no 5 or 6, all of which are valid.

Then, we will find the expected number of 5s instead of the expected number of 6s, since it is easier to sum. At the end we can subtract this from the expected number of 5s and 6s which we've already computed. We can sum with respect to the position of the first 6: there are 4^{k-1} valid sequences of length k with the first 6 occurring in position k (since both 5 and 6 cannot occur in the first $k-1$ numbers). The total number of 5s possible after those sequences is $(N-k)\cdot 6^{N-k-1}$ (by the same counting we used to count the total number above), so we get the sum

$$\sum_{k=1}^{N-1} 4^{k-1}\cdot(N-k)\cdot 6^{N-k-1}.$$

We can simplify this sum as follows:

$$\begin{aligned}
\sum_{k=1}^{N-1} 4^{k-1}\cdot(N-k)\cdot 6^{N-k-1} &= \frac{6^{N-1}}{4}\cdot\sum_{k=1}^{N-1}\frac{2^k}{3^k}(N-k)\\
&= \frac{6^{N-1}}{4}\sum_{k=1}^{N-1}\sum_{l=1}^{k}\frac{2^l}{3^l}\\
&= \frac{6^{N-1}}{4}\sum_{k=1}^{N-1}\frac{\frac{2}{3}\left(1-\frac{2^k}{3^k}\right)}{1-\frac{2}{3}}\\
&= \frac{6^{N-1}}{4}\sum_{k=1}^{N-1}2\left(1-\frac{2^k}{3^k}\right)\\
&= \frac{6^{N-1}}{2}\left(N-1-\frac{\frac{2}{3}\left(1-\frac{2^{N-1}}{3^{N-1}}\right)}{1-\frac{2}{3}}\right)\\
&= \frac{6^{N-1}}{2}\left(N-1-2\left(1-\frac{2^{N-1}}{3^{N-1}}\right)\right)\\
&= \frac{N-1}{2}6^{N-1}-6^{N-1}+4^{N-1} = \frac{N-3}{2}6^{N-1}+4^{N-1}
\end{aligned}$$

as the total number of 5s in all possible sequences. The total number of valid sequences was found earlier as $\frac{6^N+4^N}{2}$, so we then divide the total number of 5s by the number of sequences, getting $\frac{(N-3)\cdot 6^{N-1}+2\cdot 4^{N-1}}{6^N+4^N}$. In order to get the expected number of 6s, we subtract this from the total number of 5s and 6s, $\frac{2N6^{N-1}}{6^N+4^N}$, getting

$$\frac{2N6^{N-1}}{6^N+4^N}-\frac{(N-3)\cdot 6^{N-1}+2\cdot 4^{N-1}}{6^N+4^N}=\frac{(N+3)\cdot 6^{N-1}-2\cdot 4^{N-1}}{6^N+4^N}=x$$

Multiplying by 6 yields $6x=\frac{(N+3)\cdot 6^N-3\cdot 4^N}{6^N+4^N}$, which when subtracted from $\lceil 6x\rceil=\frac{(N+3)\cdot 6^N+(N+3)\cdot 4^N}{6^N+4^N}$ is equal to $\frac{(N+6)4^N}{6^N+4^N}$. Now, in order to estimate the logarithm of this quantity, we consider the 4^N in the denominator to be negligible, and take $\log_5 N+6-N\log_5\frac{3}{2}$. Note that $\frac{3^4}{2^4}=\frac{81}{16}=5+\frac{1}{16}$. This enables us to compute $N\log_5\frac{3}{2}$ very accurately as $\frac{N}{4}\log_5(5+\frac{1}{16})$. Plugging in $N=640$ gives $\log_5 646-160\log_5(5+\frac{1}{16})$. Knowing that $5^4=625\approx 646$ and $160\log_5(5+\frac{1}{16})=160\log_5 5+2\log_5((1+\frac{1}{80})^{80})\approx 160+\log_5 e^2\approx 161.25$ gives a final answer of approximately

$$\log_5 646-160\log_5\left(5+\frac{1}{16}\right)\approx 4-(161.25)=\boxed{-157.25}$$

which scores a full 25 points.

If you don't know that $(1+\frac{1}{80})^{80}$ is approximately e, you could use binomial theorem to guess that $(1+\frac{1}{80})^{80}$ was approximately $1+1+\frac{79}{160}+\frac{78}{480}+\cdots\approx 1+1+\frac{1}{2}+\frac{1}{6}=\frac{8}{3}$, so that $2\log_5\frac{8}{3}=\log_5\frac{64}{9}\approx 1.25$ as well. Alternatively, you might ignore it as negligible altogether and get an answer of $\boxed{-156}$, still scoring 21 points.

If you forget to add $\log_5 646$ (or treat the 4^N in the numerator as negligible when subtracting, which would also lead to a similar answer), you'll get a sum of $\boxed{-161.25}$ scoring 9 points or $\boxed{-160}$ scoring 14 points.