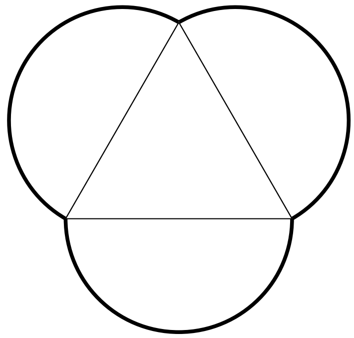
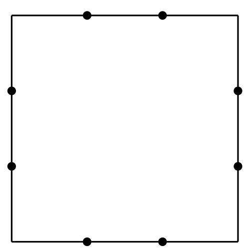
1. Andrew has three identical semicircular mooncake halves, each with radius 3, and uses them to construct the following shape, which contains an equilateral triangle in the center. Compute the perimeter around this shape, in bold below.



Answer: 9π

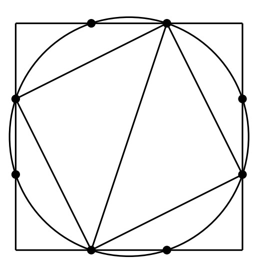
Solution: The semicircles have radii 3, so their diameters are 6. Thus, the total circumference around them is $3 \cdot \frac{6\pi}{2} = \boxed{9\pi}$.

2. On a chalkboard, Benji draws a square with side length 6. He then splits each side into 3 equal segments using 2 points for a total of 12 segments and 8 points. After trying some shapes, Benji finds that by using a circle, he can connect all 8 points together. What is the area of this circle?



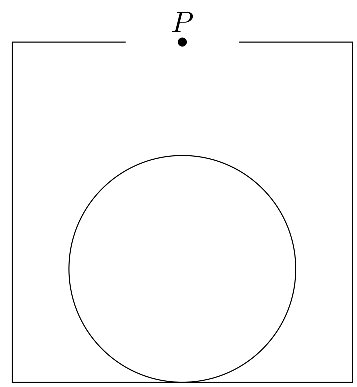
Answer: 10π

Solution: Each segment has length 2. The idea is that we only need to consider every other point to determine the circle.



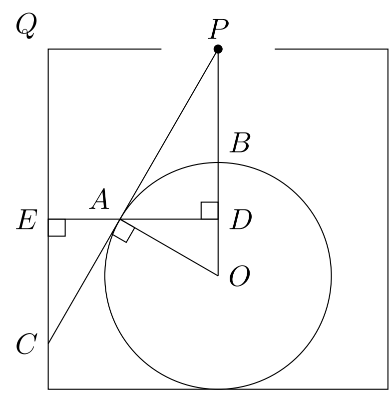
This is a square with side length $\sqrt{2^2+4^2} = 2\sqrt{5}$, and thus the diagonal has length $2\sqrt{10}$. Alternatively the diagonal of the square can be seen directly to be the hypotenuse of a triangle with legs of length 6 and 2, and so it has length $2\sqrt{10}$. The radius of the circle is half this, and so the area of the circle is $(\sqrt{10})^2\pi = \boxed{10\pi}$.

3. A square with side length 6 has a circle with radius 2 inside of it, with the centers of the square and circle vertically aligned. Aarush is standing 4 units directly above the center of the circle, at point P. What is the area of the region inside the square that he can see? (Assume that Aarush can only see parts of the square along straight lines of sight from P that are unblocked by any other objects.)



Answer: $\mathbf{13\sqrt{3} - \frac{4\pi}{3}}$

Solution:



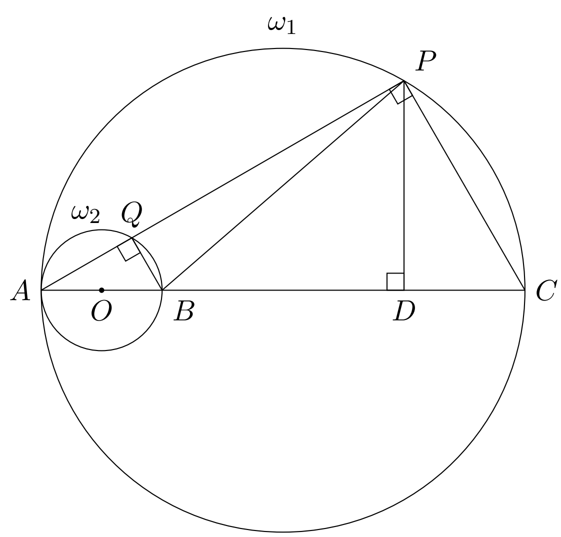
Let O be the center of the circle, let A be the left tangency point of P to the circle. Let B be the intersection of $\overline{OP}$ with the circle. Let C be the intersection of $\overline{AP}$ with the square. Let $\overline{AD}$ be the altitude from A to $\overline{OP}$. Let E be the intersection of line $\overline{AD}$ with the square on the left side. Finally, let Q be the upper left corner of the square. Our goal will be to find the area on the left side of the square, and multiply by 2. We see that the desired area is equal to the area of $\triangle CQP$ plus the area of $\triangle PAO$ minus the area of the sector of the circle under minor arc $\widehat{AB}$.

We have $OA = 2, OP = 4$ from givens, and since $\overline{OA} \perp \overline{AP}$, we can conclude that $AP = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. So, $\triangle APO$ is a 30-60-90 triangle, which gives us a lot of similar triangles. We have that $\triangle DPA \sim \triangle APO$ (they are right and share one angle), and further $\triangle DPA \sim \triangle ECA$ by vertical angles. Also, $\triangle ECA \sim \triangle QCP$ as they share an angle and are right. Therefore, $\triangle APO \sim \triangle QCP$, with their similarity ratio being $\frac{QP}{AO} = \frac{3}{2}$. We have that the area of $\triangle APO$ is equal to $\frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$, so the area of $\triangle QCP$ is equal to $\frac{9}{4} \cdot 2\sqrt{3} = \frac{9}{2}\sqrt{3}$. Then, we subtract the area of the sector of the circle under minor arc $\widehat{AB}$. Since $\triangle APO$ is a 30-60-90 triangle, we know that $\angle AOB = 60^\circ$ and so the area of the sector is $\frac{1}{6} \cdot \pi \cdot 2^2 = \frac{2\pi}{3}$. Thus, the area on the left side is $\frac{13}{2}\sqrt{3} - \frac{2\pi}{3}$, and doubling this yields the answer of $\boxed{13\sqrt{3} - \frac{4\pi}{3}}$.

4. Two circles, ω_1 and ω_2, are internally tangent at A. Let B be the point on ω_2 opposite of A. The radius of ω_1 is 4 times the radius of ω_2. Point P is chosen on the circumference of ω_1 such that the ratio $\frac{AP}{BP} = \frac{2\sqrt{3}}{\sqrt{7}}$. Let O denote the center of ω_2. Determine the ratio $\frac{OP}{AO}$.

Answer: $\mathbf{\sqrt{37}}$

Solution 1:



We can extend $\overline{AB}$ to point C on ω_1 and draw $\overline{CP}$. Let Q be the intersection of $\overline{AP}$ with ω_2. We note that $\triangle AQB \sim \triangle APC$ with a ratio of $\frac{1}{4}$, as they share an angle at A and a right angle at Q and P (since they are triangles with one side as the diameter of the circumcircle). Thus, $AQ = \frac{1}{4}AP$ and so $\frac{AQ}{BP} = \frac{1}{4} \cdot \frac{2\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3}}{2\sqrt{7}}$, and $PQ = AP - AQ = \frac{3\sqrt{3}}{2\sqrt{7}}$. Since $\triangle BQP$ is a right triangle with hypotenuse BP, we see that $PQ^2 + BQ^2 = BP^2$. Dividing by BP^2 yields $\frac{(3\sqrt{3})^2}{(2\sqrt{7})^2} + \frac{BQ^2}{BP^2} = 1$, simplifying to $\frac{BQ}{BP} = \frac{1}{2\sqrt{7}}$. This means $\triangle AQB$ has $\frac{AQ}{BQ} = \sqrt{3}$, and so $\triangle AQB$ is a 30-60-90 triangle. By similarity, $\triangle APC$ is also a 30-60-90 right triangle.

From here, we can set $AO = 1$ (since we only are given ratios of lengths, we can pick AO to make things easier). Then $\triangle APC$ has $AP = 4\sqrt{3}$, $CP = 4$, $AC = 8$. The triangle has area $\frac{1}{2} \cdot AP \cdot PC = 8\sqrt{3}$. Then, letting $\overline{PD}$ be the altitude from P to $\overline{AC}$, we have that $\frac{1}{2} \cdot AC \cdot PD = 8\sqrt{3}$, and so $PD = 2\sqrt{3}$. By the similarity of $\triangle CDP$ to $\triangle CPA$ (they are both 30-60-90 right triangles), we have that $DP = 2$ and so $OD = 5$, and therefore $OP = \sqrt{OD^2 + DP^2} = \sqrt{25 + 12} = \sqrt{37}$, and since $AO = 1$ we have that $\frac{OP}{AO} = \boxed{\sqrt{37}}$.

Solution 2: Let K be the center of ω_1. We assign coordinates (setting the radius of ω_2, AO, to 1): $A = (-4, 0), O = (-3, 0), B = (-2, 0), K = (0, 0), P = (x, y)$. Then we have the two equations $x^2 + y^2 = 16$, $\frac{(x+4)^2+y^2}{(x+2)^2+y^2} = \frac{12}{7}$ by the conditions that P is on ω_1 and $\frac{AP}{BP} = \frac{2\sqrt{3}}{\sqrt{7}}$. We can then substitute $x^2 + y^2 = 16$ on the top and bottom of both of these equations to get

$\frac{16+(8x+16)}{16+(4x+4)} = \frac{12}{7}$, which we can solve for x quickly:

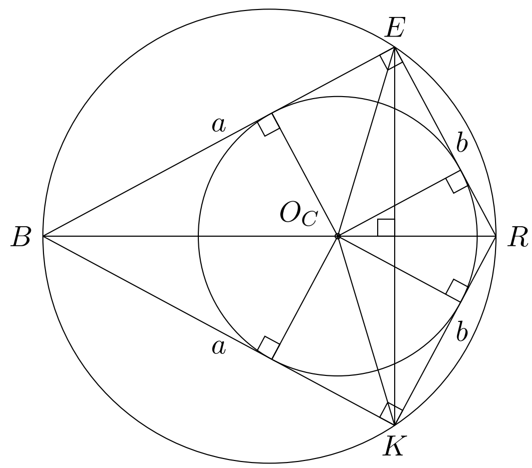
$$\frac{16+(8x+16)}{16+(4x+4)} = \frac{12}{7}$$
$$\frac{8x+32}{4x+20} = \frac{12}{7}$$
$$56x+224 = 48x+240$$
$$x = 2$$

Plugging this into $x^2+y^2=16$ gives $y=2\sqrt{3}$, and then we can compute $OP^2 = (2-(-3))^2 + (2\sqrt{3}-0)^2 = 37$, and $OP = \sqrt{37}$. Thus $\frac{OP}{AO} = \frac{\sqrt{37}}{1} = \boxed{\sqrt{37}}$.

5. Let U and C be two circles, and kite $BERK$ have vertices that lie on U and sides that are tangent to C. Given that the diagonals of the kite measure 5 and 6, find the ratio of the area of U to the area of C.

Answer: $\mathbf{\frac{66}{25}}$

Solution:



Note: there are multiple possible diagrams of $BERK$, but the final ratio of areas remains the same.

Let $BERK$ have side lengths a, a, b, b for some a and b. The kite's longer diagonal is a diameter of U due to the symmetrical nature of a kite, so U has radius $r_U = 3$ and area $A_U = 9\pi$. Moreover, splitting the kite in half shows that triangles $\triangle BER, \triangle BKR$ with side lengths a, b, 6 are right, so it follows from the Pythagorean theorem that

$$a^2+b^2 = 6^2 = 36.$$

Then, notice that $\triangle BER$ has base $BR = 6$ and height $\frac{EK}{2} = \frac{5}{2}$. Computing the area of $\triangle BER$ two ways gives

$$\frac{1}{2}ab = \frac{1}{2}\cdot\frac{5}{2}\cdot 6 = \frac{15}{2}$$

or $ab = 15$. Combining these two equations yields

$$(a+b)^2 = a^2+2ab+b^2 = 36+2(15) = 66.$$

Let O_C denote the center of incircle C, r_C denote its radius, and $[\cdot]$ denote area. Then

$$[BERK] = [BO_CE] + [EO_CR] + [RO_CK] + [KO_CB] = 2\left(\frac{ar_C}{2} + \frac{br_C}{2}\right) = (a+b)r_C.$$

Since $[BERK] = 2 \cdot \frac{1}{2}ab = 15$, we conclude that

$$A_C = r_C^2\pi = \left(\frac{15}{a+b}\right)^2 \pi = \frac{225}{66}\pi = \frac{75}{22}\pi,$$

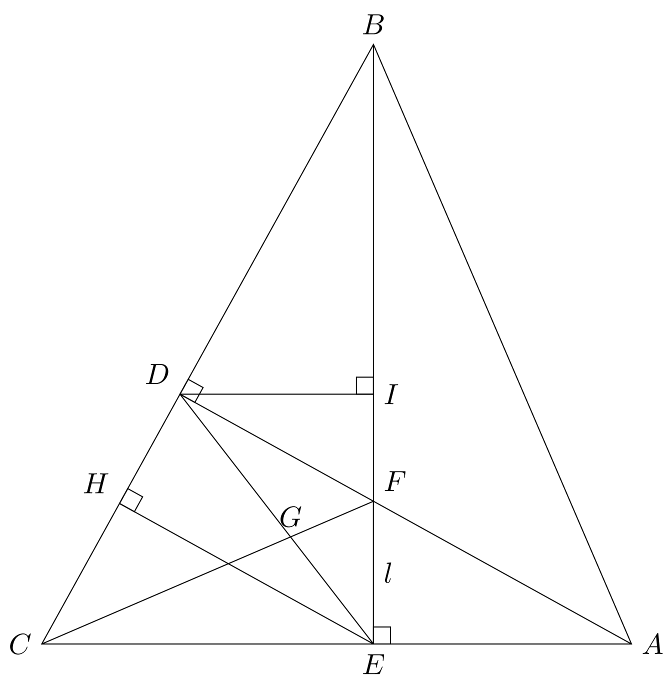
and the ratio of areas is

$$\frac{A_U}{A_C} = \frac{9\pi}{\frac{75}{22}\pi} = \boxed{\frac{66}{25}}.$$

6. Let triangle $\triangle ABC$ be acute. Point D is the foot of the altitude of $\triangle ABC$ from A to $\overline{BC}$, and E is the foot of the altitude of $\triangle ABC$ from B to $\overline{AC}$. Let F denote the point of intersection between $\overline{BE}$ and $\overline{AD}$, and let G denote the point of intersection between $\overline{CF}$ and $\overline{DE}$. The areas of triangles $\triangle EFG$, $\triangle CDG$, and $\triangle CEG$ are 1, 4, and 3 respectively. Find the area of $\triangle ABC$.

Answer: $\mathbf{\frac{448}{15}}$

Solution 1:



Since $\angle FDC = \angle FEC = 90°$, quadrilateral $CEFD$ is cyclic. Thus, $\angle FEG = \angle DCG$ by the inscribed angle theorem on arc $\widehat{DF}$ of the circumcircle of $CEFD$. So, $\triangle FEG \sim \triangle DCG$ by AA similarity. From here, let $EF = l$. Since $\frac{[CDG]}{[EFG]} = 4$, this implies that $CD = 2l$ and $\frac{DG}{FG} = 2$. We also have that $\triangle CEG \sim \triangle DFG$ (by using inscribed angle theorem on $\widehat{CD}$). Since $\frac{[CEG]}{[FEG]} = 3$ implies that $\frac{CG}{FG} = 3$ (by collinear bases and equal heights), we have $\frac{CG}{DG} = \frac{3}{2}$, and thus $[DFG] = [CEG] \cdot \left(\frac{2}{3}\right)^2 = \frac{4}{3}$.

Let H denote the foot of the altitude of triangle $\triangle CDE$ from E to $\overleftrightarrow{CD}$, and let I denote the foot of the altitude of triangle $\triangle DEF$ from D to $\overleftrightarrow{EF}$. In this case, $\triangle BDF \sim \triangle BHE$ and $\triangle BDI \sim \triangle BCE$. For the scale factors of the similar triangles, $\frac{EH}{DF} = \frac{[CDE]}{[CDF]} = \frac{7}{16/3} = \frac{21}{16}$ and $\frac{CE}{DI} = \frac{[CEF]}{[DEF]} = \frac{4}{7/3} = \frac{12}{7}$. By triangle similarity, this implies that $BE = BF + FE = BF + l = \frac{21}{16}BF$ and $BC = BD + CD = BD + 2l = \frac{12}{7}BD$, so $BF = \frac{16}{5}l$ and $BD = \frac{14}{5}l$.

By the Pythagorean theorem, we have that $DF = \frac{2\sqrt{15}}{5}l$ and $CE = \frac{3\sqrt{15}}{5}l$. Since $\triangle BDF \sim \triangle AEF$, $\frac{EF}{FD} = \frac{AE}{BD}$, so $AE = \frac{7\sqrt{15}}{15}l$. Thus, $[ABC] = [CEF] \cdot \frac{AC}{EC} \cdot \frac{BE}{EF} = 4 \cdot \frac{16}{9} \cdot \frac{21}{5} = \boxed{\frac{448}{15}}$.

Solution 2: Proceed as in the first solution to find that $[FGD] = \frac{4}{3}$, and define H and I as in that solution. Since $\triangle BDF \sim \triangle BHE$, it follows that

$$\frac{BF}{BE} = \frac{DF}{HE} = \frac{\frac{1}{2} \cdot DF \cdot CD}{\frac{1}{2} \cdot EH \cdot CD} = \frac{[DFC]}{[DEC]} = \frac{\frac{4}{3} + 4}{3 + 4} = \frac{16}{21}.$$

Likewise, since $\triangle AEF \sim \triangle AID$,

$$\frac{AE}{AI} = \frac{EF}{ID} = \frac{[EFC]}{[DEC]} = \frac{1+3}{4+3} = \frac{4}{7}.$$

Finally, since

$$\frac{CE}{IE} = \frac{[CEF]}{[DEF]} = \frac{1+3}{1+\frac{4}{3}} = \frac{12}{7},$$

we conclude that

$$AE : EI : IC = 4 : 3 : \frac{15}{7}.$$

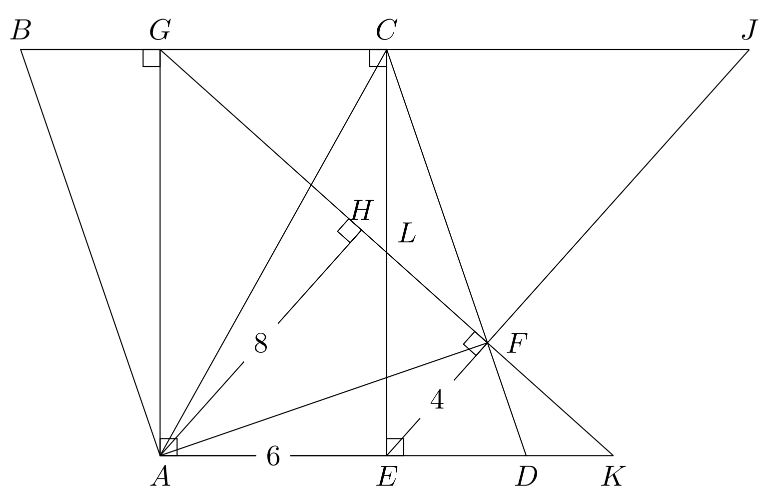
Then

$$[ABC] = [CEF] \cdot \frac{AC}{CE} \cdot \frac{BE}{EF} = (1+3) \cdot \frac{4+3+\frac{15}{7}}{3+\frac{15}{7}} \cdot \frac{21}{5} = \boxed{\frac{448}{15}}.$$

7. In parallelogram $ABCD$, E is a point on $\overline{AD}$ such that $\overline{CE} \perp \overline{AD}$, F is a point on $\overline{CD}$ such that $\overline{AF} \perp \overline{CD}$, and G is a point on $\overline{BC}$ such that $\overline{AG} \perp \overline{BC}$. Let H be a point on $\overline{GF}$ such that $\overline{AH} \perp \overline{GF}$, and let J be the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{BC}$. Given that $AH = 8$, $AE = 6$, and $EF = 4$, compute CJ.

Answer: $\mathbf{\frac{48}{5}}$

Solution 1:



Connect $\overline{AC}$. For $\triangle AFD$ and $\triangle CED$, $\angle AFD = \angle CED = 90°$, $\angle ADF = \angle CDE$, so $\triangle AFD \sim \triangle CED$, and $\frac{FD}{ED} = \frac{AD}{CD}$. So we also have $\triangle EFD \sim \triangle CAD$, and $\angle EFD = \angle CAD$. Since $\angle AGC = \angle AFC = 90°$, A, G, C, F are cyclic, $\overline{AD} \parallel \overline{BC}$, we have:

$$\angle GCA = \angle GFA = \angle CAD = \angle EFD.$$

And thus,

$$\angle AFG + \angle AFE = \angle AFE + \angle EFD = 90°.$$

So,

$$\overline{EF} \perp \overline{FG}, \overline{EF} \parallel \overline{AH}.$$

Let K be the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{GF}$, L be the intersection of $\overline{EC}$ and $\overline{FG}$, $ED = x$, $CJ = y$, $EL = z$.
Since $\overleftrightarrow{EF} \parallel \overleftrightarrow{AH}$, $\angle EKF = \angle AKH$, $\frac{EK}{AK} = \frac{EF}{AH} = \frac{1}{2}$. So $EK = 6, DK = 6 - x$. Since $\overleftrightarrow{AK} \parallel \overleftrightarrow{BJ}$,

$$\frac{x}{y} = \frac{6-x}{6}.$$

Since $\overleftrightarrow{EK} \parallel \overleftrightarrow{GC}$ and $EK = GC = 6$, $\triangle ELK \cong CLG$. So L is the midpoint of $\overline{CE}$. Since $\triangle EFL \sim \triangle ECJ$, $\frac{EF}{EL} = \frac{EC}{EJ}$, and since $\triangle EFD \sim \triangle JFC$,

$$\frac{ED}{EF} = \frac{JC}{JF},$$

$$\frac{x}{4} = \frac{y}{FJ},$$

$$\therefore FJ = 4 \cdot \frac{y}{x}.$$

Since $\triangle LEF \sim \triangle JEC$,

$$\frac{4}{z} = \frac{2z}{4 + 4 \cdot \frac{y}{x}},$$

and thus,

$$z = \sqrt{8 + 8 \cdot \frac{y}{x}},$$

$$z = \sqrt{8 + 8 \cdot \frac{6}{6-x}},$$

$$FL^2 = z^2 - 16 = 8 \cdot \frac{6}{6-x} - 8$$

Let M be a point on $\overline{GL}$ such that $\overline{CM} \perp \overline{GL}$. Since $\triangle LMC \sim \triangle CMG \sim \triangle LCG$, $\triangle CML \cong EFL$ we have

$$EF^2 = CM^2 = LM \cdot MG = FL \cdot MG.$$

Applying Pythagorean Theorem on right triangle $\triangle CMG$, $MG = \sqrt{GC^2 - MC^2} = \sqrt{6^2 - 4^2} = \sqrt{20}$. Thus,

$$4^2 = \sqrt{8 \cdot \frac{6}{6-x} - 8} \cdot \sqrt{20},$$

$$x = \frac{48}{13}.$$

Applying $x = \frac{48}{13}$ to $\frac{x}{y} = \frac{6-x}{6}$, we have $CJ = y = \frac{6 \cdot \frac{48}{13}}{6 - \frac{48}{13}} = \boxed{\frac{48}{5}}$.

Solution 2: $AECG$ is a rectangle, and is therefore cyclic. Since $\angle AEC = \angle AFC = 90°$, it follows that $AEFC$ is cyclic. Thus, $EFCG$ is cyclic, so that $\angle EFG = \angle ECG = 90°$, and $\overline{EF} \perp \overline{FG}$. Since $\overline{AH} \perp \overline{FG}$, it follows that $\overline{EF} \parallel \overline{AH}$.

Let $\overline{AE}$ and $\overline{FG}$ intersect at X. Also denote $x = JF$, $y = JC$. Then $\triangle EFX \sim \triangle AHX \sim \triangle JFG$ implies that

$$\frac{y+6}{x} = \frac{JG}{JF} = \frac{EX}{EF} = \frac{AX}{AH} = \frac{AX - EX}{AH - EF} = \frac{6}{8-4} = \frac{3}{2}.$$

Since $EFCG$ is cyclic, power of a point on J yields

$$JF \cdot JE = JC \cdot JG \implies x(x+4) = y(y+6).$$

Substituting $y = \frac{3}{2}x - 6$, we obtain

$$x(x+4) = \left(\frac{3}{2}x - 6\right) \cdot \frac{3}{2}x = x\left(\frac{9}{4}x - 9\right).$$

Solving, $x = \frac{52}{5}$ and $CJ = y = \boxed{\frac{48}{5}}$.

8. Points A, B, C, D, E, and F lie on a sphere with radius $\sqrt{10}$ such that lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent at point P inside the sphere and are pairwise perpendicular. If $PA = \sqrt{6}$, $PB = \sqrt{10}$, and $PC = \sqrt{15}$, what is the volume of tetrahedron $DEFP$?

Answer: $\mathbf{\frac{81}{20}}$

Solution: Let the power of point P with respect to the sphere be x. Then, $AP \cdot DP = BP \cdot EP = CP \cdot FP = x$, so $DP = \frac{x}{\sqrt{6}}$, $EP = \frac{x}{\sqrt{10}}$, and $FP = \frac{x}{\sqrt{15}}$.

Let O be the center of the sphere. We now find the distance OP by drawing the perpendicular lines from O onto $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$; label these intersections Q, R, and S, respectively. Since the perpendicular lines from the center of a sphere onto a chord bisects the chord, we have $AQ = QD$, $BR = RE$, and $CS = SF$, so $QP = |\frac{\sqrt{6}}{2} - \frac{x}{2\sqrt{6}}|$, $RP = |\frac{\sqrt{10}}{2} - \frac{x}{2\sqrt{10}}|$, and $SP = |\frac{\sqrt{15}}{2} - \frac{x}{2\sqrt{15}}|$.

Due to the right angles given by the perpendicular lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$, P, Q, R, and S are vertices of a rectangular prism, where the neighboring vertices of P are Q, R, and S. Furthermore, due to the right angles created by dropping the perpendiculars from O onto $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, O is also a vertex of this rectangular prism, and is the opposite vertex from P. By the distance formula in the prism,

$$OP^2 = QP^2 + RP^2 + SP^2 = \left(\frac{\sqrt{6}}{2} - \frac{x}{2\sqrt{6}}\right)^2 + \left(\frac{\sqrt{10}}{2} - \frac{x}{2\sqrt{10}}\right)^2 + \left(\frac{\sqrt{15}}{2} - \frac{x}{2\sqrt{15}}\right)^2.$$

Then, the power of point P is $(\sqrt{10})^2 - OP^2 = 10 - OP^2$. Equating this to x gives the equation

$$10 - \left(\left(\frac{\sqrt{6}}{2} - \frac{x}{2\sqrt{6}}\right)^2 + \left(\frac{\sqrt{10}}{2} - \frac{x}{2\sqrt{10}}\right)^2 + \left(\frac{\sqrt{15}}{2} - \frac{x}{2\sqrt{15}}\right)^2\right) = x.$$

Simplifying the left-hand side gives

$$-\frac{1}{12}x^2 + \frac{3}{2}x + \frac{9}{4} = x.$$

This simplifies to the quadratic equation $x^2 - 6x + 27 = 0$, which factors as $(x-9)(x+3) = 0$, so $x = 9$ or $x = -3$. Since we defined the power of point P to be positive, we discard the negative solution and conclude that $x = 9$.

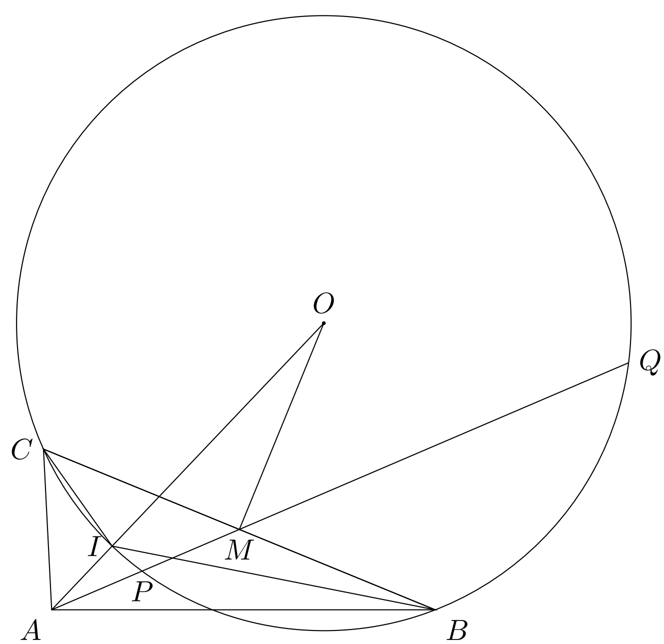
Finally, the volume of tetrahedron $DEFP$ is

$$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{x}{PA} \cdot \frac{x}{PB} \cdot \frac{x}{PC} = \frac{x^3}{6PA \cdot PB \cdot PC} = \frac{9^3}{6 \cdot \sqrt{6} \cdot \sqrt{10} \cdot \sqrt{15}} = \frac{9^3}{6 \cdot 30} = \boxed{\frac{81}{20}}.$$

9. Let $\triangle ABC$ be a triangle with incenter I, and let M be the midpoint of $\overline{BC}$. Line $\overleftrightarrow{AM}$ intersects the circumcircle of triangle $\triangle IBC$ at points P and Q. Suppose that $AP = 13, AQ = 83$, and $BC = 56$. Find the perimeter of $\triangle ABC$.

   Answer: 128

   Solution:



Let ω be the circumcircle of $\triangle IBC$ with center O. First, note that by the incenter-excenter lemma, we have that the circumcenter of $\triangle IBC$ is on $\overrightarrow{AI}$. Then, without loss of generality, we assume $AB \geq AC$. We begin by showing that $AB = AC$ is impossible. If $AB = AC$, I is on $\overline{AM}$ and so $I = P$. This gives $AI = 13$ and therefore $IM = MP < 13$ as the incenter's perpendicular distances to $\overline{AB}, \overline{BC}, \overline{AC}$ are equal and less than the distance to any other points on those lines. However, by power of a point on M with respect to ω we have $BM \cdot CM = PM \cdot QM$, and since $PQ = 83 - 13 = 70$ and $BM \cdot CM = 28^2$, we have $PM, QM = 14, 56$ in some order. This would imply $PM > 13$ but we already proved that $PM = IM = 13$ which is a contradiction.

So, we have that $AB > AC$. In this case, power of a point on A with respect to ω gives $AB \cdot AC = AP \cdot AQ$, as the second intersection of $\overrightarrow{AC}$ with ω is at the reflection of B over the angle bisector

$\overrightarrow{AI}$. Therefore, we have $AB \cdot AC = 13 \cdot 83 = 1079$. Like in the isosceles case, we apply power of a point to M and get that $PM, QM = 14, 56$ in some order. Since $\overline{OM} \perp \overline{BC}$ (perpendicular bisectors intersect at the circumcenter), and A is on the other side of $\overline{BC}$ from O, we know that $\angle OMA$ is obtuse. This implies that $PM < QM$, so $PM = 14, AM = AP + PM = 27$.

We can now apply Stewart's theorem on $\triangle ABC$ with point M to get

$$AB^2\left(\frac{BC}{2}\right) + AC^2\left(\frac{BC}{2}\right) = \frac{BC^3}{4} + BC \cdot AM^2$$

and plug in BC and AM, which reduces to $AB^2 + AC^2 = 2(28^2 + 27^2) = 3026$. Using $AB \cdot AC = 1079$ that we found earlier, we get that

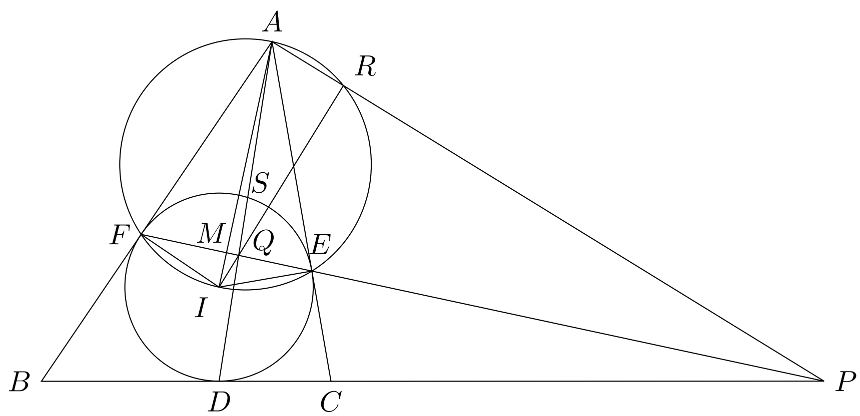
$$AB^2 + AC^2 + 2AB \cdot AC = (AB + AC)^2 = 3026 + 2(1079) = 5184 = 72^2$$

which tells us that $(AB + AC) + BC = 72 + 56 = \boxed{128}$.

10. The incircle of scalene triangle $\triangle ABC$ is tangent to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at points D, E, and F, respectively. The line $\overleftrightarrow{EF}$ intersects line $\overleftrightarrow{BC}$ at P and line $\overleftrightarrow{AD}$ at Q. The circumcircle of $\triangle AEF$ intersects line $\overleftrightarrow{AP}$ again at point $R \neq A$. If $QE = 3$, $QF = 4$, and $QR = 8$, find the area of triangle $\triangle AEF$.

Answer: $\frac{343\sqrt{2}}{16}$

Solution:



Let $\overline{AD}$ intersect the incircle again at S. Then since $\overline{AD}$ is a symmedian of $\triangle DEF$, quadrilateral $DESF$ is harmonic. Therefore, the tangent at D, the tangent at S, and line EF are concurrent, and since $P = \overline{BC} \cap \overline{EF}$, P lies on the tangent at S as well.

Henceforth, we will refer to poles, polars, and inverses with respect to the incircle of $\triangle ABC$. Since $\overline{DS}$ is the chord of contact of point P, $\overline{DS}$ is the polar of P. Because Q lies on $\overline{DS}$, by La Hire's theorem P lies on the polar of Q. Similarly, since $\overline{EF}$ is the chord of contact of point A, $\overleftrightarrow{EF}$ is the polar of A. Because Q lies on $\overline{EF}$, by La Hire's theorem A lies on the polar of Q. Since A and P are distinct points on the polar of Q, $\overleftrightarrow{AP}$ is the polar of Q.

Let I be the incenter of $\triangle ABC$. Then $\overline{AI}$ is a diameter of the circumcircle of $\triangle AEF$ since $\angle IEA = \angle IFA = 90°$. Then $\angle IRA = 90°$ as well, so $\overline{IR} \perp \overline{AP}$. But since R lies on $\overline{AP}$, which is the polar of Q, we get that R is the inverse of Q.

Let the inradius be r and let $QI = k$. Then $QR = RI - QI = \frac{r^2}{k} - k = \frac{r^2 - k^2}{k}$ and $QE \cdot QF = r^2 - k^2$, so we get the system of equations

$$\frac{r^2 - k^2}{k} = 8$$
$$r^2 - k^2 = 12$$

Dividing the second equation by the first gives $k = \frac{3}{2}$, so $r = \sqrt{12 + k^2} = \frac{\sqrt{57}}{2}$.

To find the area of $\triangle AEF$, draw the perpendicular bisector $\overline{AI}$ of $\overline{EF}$, letting the point of intersection be M. Then $EM = \frac{1}{2}EF = \frac{7}{2}$, and $MI = \sqrt{r^2 - EM^2} = \sqrt{\frac{57}{4} - \frac{49}{4}} = \sqrt{2}$. Since $\triangle AME \sim \triangle EMI$ by AA similarity, we get $\frac{AM}{EM} = \frac{EM}{IM}$, so $AM = \frac{(7/2)^2}{\sqrt{2}} = \frac{49\sqrt{2}}{8}$. Finally, the area of $\triangle AEF$ is $\frac{1}{2} \cdot AM \cdot EF = \frac{1}{2} \cdot \frac{49\sqrt{2}}{8} \cdot 7 = \boxed{\frac{343\sqrt{2}}{16}}$.