1. Compute
$$2 + 0 + 2 + 4 + 2 \times 0 + 2 \times 2 + 2 \times 4 + 0 \times 2 + 0 \times 4 + 2 \times 4.$$

Answer: 28

Solution: The products with a 0 in them evaluate to 0, so we have

$$2+0+2+4+2\cdot 0+2\cdot 2+2\cdot 4+0\cdot 2+0\cdot 4+2\cdot 4 = 2+2+4+2\cdot 2+2\cdot 4+2\cdot 4 = 8+4+8+8 = 28,$$

so the answer is $\boxed{28}$

2. When the odd positive two-digit number 11 is added to 46, the result is 57, whose sum of digits is $5 + 7 = 12$.

What odd positive two-digit number can be added to 46 so the result is a number whose digits sum to 17?

Answer: 43

Solution: The two-digit numbers with a sum of digits of 17 are 89 and 98. The least three-digit number with a sum of digits of 17 is 179, but since $179 - 46 = 133$ has more than two digits, this sum is not attainable.

So, the desired number is either $89 - 46 = 43$ or $98 - 46 = 52$. Between these two possibilities, the only odd number is $\boxed{43}$.

3. At a certain point in time, Nikhil had 3 more apples than Brian. Theo then gave Nikhil 9 apples and took away 3 apples from Brian. Now, Nikhil has twice as many apples as Brian. Compute the number of apples that Nikhil and Brian now have in total.

Answer: 45

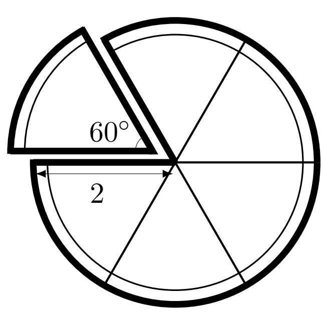
Solution: Suppose Nikhil starts with N apples and Brian with B apples. We then have

$$N = B + 3 \quad (1)$$

$$N + 9 = 2(B - 3) \quad (2)$$

based on the given information. Substituting (1) into (2) gives $B + 12 = 2(B - 3)$, so $B + 12 = 2B - 6$, and $B = 18$. So, $N = 21$. Our final answer is then $(N + 9) + (B - 3) = 30 + 15 = \boxed{45}$.

4. Jessica cuts a pie into perfect sixths. If the pie's radius is 2, what is the perimeter of one slice of the pie?



Answer: $\mathbf{4+\frac{2\pi}{3}}$

Solution: The two straight sides of the slice have length equal to the radius, and the round side has length equal to one sixth of the circumference of the whole pie. Therefore, our answer is $2\cdot 2+\frac{1}{6}\cdot 2\pi\cdot 2=\boxed{4+\frac{2\pi}{3}}$.

5. Find the smallest number among the following numbers:

$$\frac{7}{13}, \frac{10}{19}, \frac{5}{9}, \frac{3}{5}, \frac{2023}{4045}, \frac{6}{11}, \frac{2024}{4047}, \frac{4}{7}.$$

Answer: $\mathbf{\frac{2024}{4047}}$

Solution: Intuitively, these fractions are all just a little over $\frac{1}{2}$ since their denominators are just under twice their numerators. More rigorously, the fractions are all of the form $\frac{x}{2x-1}$. We can rewrite this as $\frac{1}{2}+\frac{1}{4x-2}$. This is minimized when x is the greatest, and thus the answer is $\boxed{\frac{2024}{4047}}$

6. Find the third-largest three-digit multiple of three that is a palindrome. (Recall that a palindrome is a number that reads the same forward and backward, such as 444 or 838, but not 227.)

Answer: 939

Solution: In order for a three-digit number to be a palindrome, its units and hundreds digit must be equal, and for a number to be a multiple of three its digits must sum to a multiple of three. So, a number of the form $\underline{9}\underline{A}\underline{9}$ must have that $9+A+9=18+A$ is a multiple of three, which means that A is a multiple of three. Therefore, the largest three-digit multiple of three that is a palindrome is 999, the second largest is 969, and the third largest is $\boxed{939}$.

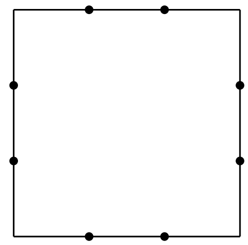
7. Café BMT makes their famous Bacon, Mozzarella, Tomato sandwich with 3 strips of bacon, 1 slice of mozzarella, and 1 tomato slice stacked on top of each other. How many ways can the toppings of a Bacon, Mozzarella, Tomato sandwich be arranged?

Answer: 20

Solution: The number of ways of ordering the toppings can be counted by pretending all of the toppings are distinct, ordering them, and then dividing out by the overcounted cases. In this instance, we order the 5 toppings in 5! ways, and then we overcount each case 3! times because the 3 strips of bacon are indistinguishable. So, any of the 3! different orderings of the bacon in any given arrangement are all identical. (The same logic applies to the mozzarella and tomato, but since there are only 1 of each of those there is only one way of ordering them.) So, we have that the number of orderings is:

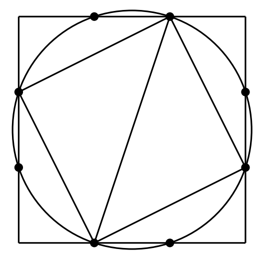
$$\frac{5!}{1!\cdot 3!\cdot 1!}=5\cdot 4=\boxed{20}$$

8. On a chalkboard, Benji draws a square with side length 6. He then splits each side into 3 equal segments using 2 points for a total of 12 segments and 8 points. After trying some shapes, Benji finds that by using a circle, he can connect all 8 points together. What is the area of this circle?



Answer: 10π

Solution: Each segment has length 2. The idea is that we only need to consider every other point to determine the circle.



This is a square with side length $\sqrt{2^2+4^2} = 2\sqrt{5}$, and thus the diagonal has length $2\sqrt{10}$. Alternatively the diagonal of the square can be seen directly to be the hypotenuse of a triangle with legs of length 6 and 2, and so it has length $2\sqrt{10}$. The radius of the circle is half this, and so the area of the circle is $(\sqrt{10})^2\pi = \boxed{10\pi}$.

9. Find the real number x satisfying

$$\frac{x^2-20}{x^2+20x+4} = \frac{x^2-24}{x^2+24x+4} = \frac{1}{2}.$$

Answer: −2

Solution: Working with $\frac{x^2-20}{x^2+20x+4} = \frac{1}{2}$, we can cross multiply and simplify as follows:

$$2x^2 - 40 = x^2 + 20x + 4$$
$$x^2 - 20x - 44 = 0$$
$$(x-22)(x+2) = 0.$$

Therefore, $x = 22$ or $x = -2$.

Next, we can solve $\frac{x^2-24}{x^2+24x+4} = \frac{1}{2}$ by cross multiplying:

$$2x^2 - 48 = x^2 + 24x + 4$$
$$x^2 - 24x - 52 = 0$$
$$(x-26)(x+2) = 0.$$

Here, $x = 26$ or $x = -2$. The only solution to both equations is therefore $x = \boxed{-2}$.

10. Suppose $a_1, a_2, \ldots$ is an arithmetic sequence, and suppose $g_1, g_2, \ldots$ is a geometric sequence with common ratio 2. Suppose $a_1 + g_1 = 1$ and $a_2 + g_2 = 1$. If $a_{24} = g_7$, find a_{2024}.

    Answer: -22

    Solution: Let $g_1 = S$ and $g_2 = 2S$. Then we have $a_1 = 1 - S$ and $a_2 = 1 - 2S$. Suppose the common difference of the arithmetic sequence is d. This means $d = 1 - 2S - (1 - S) = -S$. Now we can solve for $a_{24} = g_7$ as follows:

$$\begin{aligned} g_7 &= 2^6 S = 64S \\ a_{24} &= a_1 + 23d = (1 - S) - 23S = 1 - 24S \\ 64S &= 1 - 24S \\ S &= \frac{1}{88} \end{aligned}$$

    Therefore,

$$a_{2024} = a_1 + 2023d = 1 - S + 2023(-S) = 1 - 2024S = 1 - 2024\left(\frac{1}{88}\right) = 1 - 23 = \boxed{-22}.$$

11. The 35-step staircase of Sather Tower is being renovated. Each step will be painted a single color such that the stairs repeat color every 5 steps. There are 14 available stair colors, including blue and gold. Each color may only cover up to 10 steps, but both blue and gold must be used. With these restrictions, in how many different ways can the stairs be colored?

    Answer: 26400

    Solution: Since the pattern of 5 steps repeats 7 times, the number of stairs of any given color must be a multiple of 7. But there is only enough paint for up to 10 stairs of one color, so each color can be used at most once in the pattern. There are 5×4 ways to choose which stairs in the pattern receive the blue and gold colors. Then, for the remaining 3 stairs in the pattern, there are $\frac{12!}{9!} = 12 \times 11 \times 10$ ways to color them. So, the total number of ways is

$$5 \times 4 \times 12 \times 11 \times 10 = 20 \times 132 \times 10 = \boxed{26400}.$$

12. Find the greatest integer less than

$$\frac{2}{1} - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \cdots - \frac{2}{2024}.$$

    Answer: 1

    Solution: Let x be the expression. We first factor out the 2 and consider the expression

$$\frac{x}{2} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots - \frac{1}{2024}.$$

    We can bound this expression from below by noting that $\frac{1}{1} - \frac{1}{2} = \frac{1}{2}$ and every subsequent pair of fractions is positive as $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0$:

$$\frac{x}{2} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2023} - \frac{1}{2024}\right).$$

    Similarly, we can bound this expression from above by noting that $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and every subsequent pair of fractions is negative: $-\frac{1}{n} + \frac{1}{n+1} = -\frac{1}{n(n+1)} < 0$. This leaves a last term of $-\frac{1}{2024}$, but this is also negative, so $\frac{5}{6}$ is an upper bound on the expression.
    Thus, $\frac{1}{2} < \frac{x}{2} < \frac{5}{6}$, or $1 < x < \frac{5}{3}$. Since $x > 1$ and $x < \frac{5}{3}$, the greatest integer less than x is $\boxed{1}$.

13. Find the number of positive integers, n, such that $\dfrac{20+n}{24-n}$ is an integer.

Answer: 11

Solution: We are given that $\dfrac{20+n}{24-n}$ is an integer. We can simplify the numerator as follows:

$$\frac{20+n}{24-n} = \frac{20+n}{24-n} + \frac{24-n}{24-n} - 1 = \frac{20+n+24-n}{24-n} - 1 = \frac{44}{24-n} - 1.$$

So, we want to determine when $\dfrac{44}{24-n}$ is an integer. This occurs whenever $24-n$ is a factor of 44. Since n is positive, $24-n$ is less than 24, so the possible values of $24-n$ are the positive and negative divisors of 44 that are less than 24. These include $\pm1, \pm2, \pm4, \pm11, \pm22, -44$. We conclude that there are $\boxed{11}$ possible values of n.

14. How many terms of the sequence $3^1+1, 3^2+2, 3^3+3, \ldots, 3^{2024}+2024$ are divisible by 5?

Answer: 406

Solution: The sequence of numbers modulo 5 repeats every 20 since modulo 5, the terms 3^i cycle every 4 terms $(3, 4, 2, 1, 3, \ldots)$ and the terms i cycle every 5 terms $(1, 2, 3, 4, 0, 1, \ldots)$.

For each $k = 1, 2, 3, 4$, consider the sequence created by taking every fourth term of the sequence starting with $3^k + k$. The 3^k part remains the same modulo 5, so these terms are equivalent to $3^k+k, 3^k+k+4, 3^k+k+8, 3^k+k+12, 3^k+k+16, 3^k+k+20, \ldots$ modulo 5, which cycles every 5 terms. Since $0, 4, 8, 12, 16$ take up each residue modulo 5, one in every 5 consecutive terms is divisible by 5. Combining the four sequences from $k = 1, 2, 3, 4$ together, we get that for every 20 consecutive terms in the whole sequence, exactly 4 of them are divisible by 5.

For the terms $3^5+5, 3^6+6, \ldots, 3^{2024}+2024$ we calculate that $\frac{4}{20} \cdot 2020 = 404$ of them are divisible by 5. Then, we check that $3^1+1 \equiv 4$, $3^2+2 \equiv 1$, $3^3+3 \equiv 0$, and $3^4+4 \equiv 0$ all modulo 5, so the answer is $404 + 2 = \boxed{406}$.

15. For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n and let $\lceil n \rceil$ be the smallest integer greater than or equal to n. For example, $\lfloor 2.5 \rfloor = 2$ and $\lfloor 2 \rfloor = 2$, while $\lceil 2.5 \rceil = 3$ and $\lceil 2 \rceil = 2$. Find the greatest integer x such that $\left\lfloor \frac{x}{20} + 20 \right\rfloor = \left\lceil \frac{x}{24} + 24 \right\rceil$.

Answer: 699

Solution: We can simplify the equation as follows:

$$\left\lfloor \frac{x}{20} \right\rfloor + 20 = \left\lceil \frac{x}{24} \right\rceil + 24$$

$$\left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil = 4.$$

Note that $a \le \lceil a \rceil < a+1$ for any real a, and similarly $a - 1 < \lfloor a \rfloor \le a$ as well. We then have that

$$\frac{x}{20} - 1 - \left(\frac{x}{24} + 1\right) < \left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil = 4 \le \frac{x}{20} - \frac{x}{24}$$

yielding the bounds

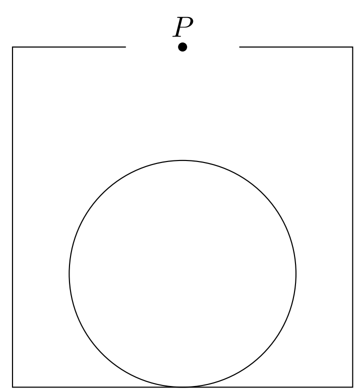
$$4 \le \frac{x}{20} - \frac{x}{24} < 6$$

Since $\frac{x}{20} - \frac{x}{24} = \frac{x}{120}$, we have $480 \le x < 720$. We can now go backwards from 720 to find the solution. We can decrease $\left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil$ by 1 by decreasing x by 1, which will reduce the floor expression by 1. To decrease by another increment of 1, we will need to decrease x by another 20. Therefore, $x = 720 - 20 - 1 = 699$ gives us

$$\left\lfloor \frac{699}{20} \right\rfloor - \left\lceil \frac{699}{24} \right\rceil = 34 - 30 = 4$$

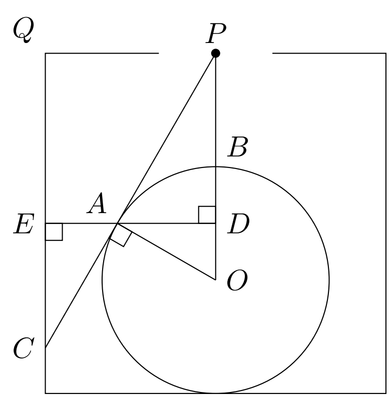
and our answer is $\boxed{699}$.

16. A square with side length 6 has a circle with radius 2 inside of it, with the centers of the square and circle vertically aligned. Aarush is standing 4 units directly above the center of the circle, at point P. What is the area of the region inside the square that he can see? (Assume that Aarush can only see parts of the square along straight lines of sight from P that are unblocked by any other objects.)



Answer: $\mathbf{13\sqrt{3} - \frac{4\pi}{3}}$

Solution:



Let O be the center of the circle, let A be the left tangency point of P to the circle. Let B be the intersection of $\overline{OP}$ with the circle. Let C be the intersection of $\overline{AP}$ with the square. Let $\overline{AD}$ be the altitude from A to $\overline{OP}$. Let E be the intersection of line $\overline{AD}$ with the square on the left side. Finally, let Q be the upper left corner of the square. Our goal will be to find the area on the left side of the square, and multiply by 2. We see that the desired area is equal to the area of $\triangle CQP$ plus the area of $\triangle PAO$ minus the area of the sector of the circle under minor arc $\widehat{AB}$.

We have $OA = 2, OP = 4$ from givens, and since $\overline{OA} \perp \overline{AP}$, we can conclude that $AP = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. So, $\triangle APO$ is a 30-60-90 triangle, which gives us a lot of similar triangles. We have that $\triangle DPA \sim \triangle APO$ (they are right and share one angle), and further $\triangle DPA \sim \triangle ECA$ by vertical angles. Also, $\triangle ECA \sim \triangle QCP$ as they share an angle and are right. Therefore, $\triangle APO \sim \triangle QCP$, with their similarity ratio being $\frac{QP}{AO} = \frac{3}{2}$. We have that the area of $\triangle APO$ is equal to $\frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$, so the area of $\triangle QCP$ is equal to $\frac{9}{4} \cdot 2\sqrt{3} = \frac{9}{2}\sqrt{3}$. Then, we subtract the area of the sector of the circle under minor arc $\overset{\frown}{AB}$. Since $\triangle APO$ is a 30-60-90 triangle, we know that $\angle AOB = 60^\circ$ and so the area of the sector is $\frac{1}{6} \cdot \pi \cdot 2^2 = \frac{2\pi}{3}$. Thus, the area on the left side is $\frac{13}{2}\sqrt{3} - \frac{2\pi}{3}$, and doubling this yields the answer of $\boxed{13\sqrt{3} - \frac{4\pi}{3}}$.

17. Eight players are seated around a circular table. Each player is assigned to either Team Green or Team Yellow so that each team has at least one player. In how many ways can the players be assigned to the teams such that each player is on the same team as at least one player adjacent to them?

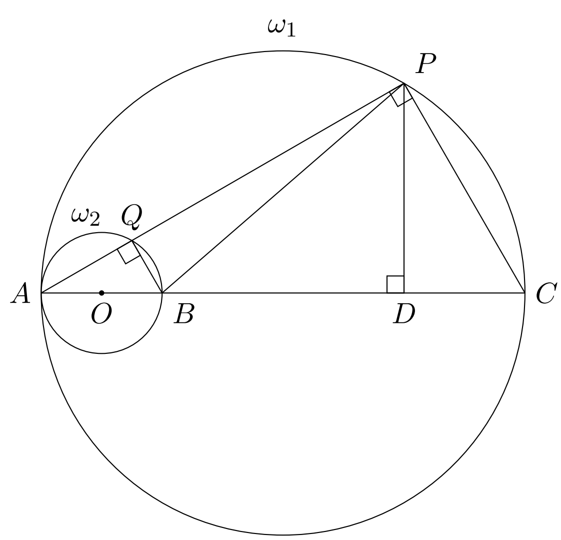
Answer: 44

Solution: Since not everyone can be on the same team, there are an even number of contiguous groups of players on the same team, alternating between Green and Yellow. There are either four groups of two players on the same team, or there are two contiguous groups on the opposing teams. In the former case, there are 4 ways (not 8, since this configuration has 180° rotational symmetry). In the latter case divides between Green and Yellow may be 2 vs. 6, 3 vs. 5, 4 vs. 4, 5 vs. 3, or 6 vs. 2, and each of these cases has 8 ways for each rotation around the table. There are a total of $4 + 8 \cdot 5 = \boxed{44}$ ways.

18. Two circles, ω_1 and ω_2, are internally tangent at A. Let B be the point on ω_2 opposite of A. The radius of ω_1 is 4 times the radius of ω_2. Point P is chosen on the circumference of ω_1 such that the ratio $\frac{AP}{BP} = \frac{2\sqrt{3}}{\sqrt{7}}$. Let O denote the center of ω_2. Determine the ratio $\frac{OP}{AO}$.

Answer: $\sqrt{37}$

Solution 1:



We can extend $\overline{AB}$ to point C on ω_1 and draw $\overline{CP}$. Let Q be the intersection of $\overline{AP}$ with ω_2. We note that $\triangle AQB \sim \triangle APC$ with a ratio of $\frac{1}{4}$, as they share an angle at A and a right angle at Q and P (since they are triangles with one side as the diameter of the circumcircle). Thus, $AQ = \frac{1}{4}AP$ and so $\frac{AQ}{BP} = \frac{1}{4} \cdot \frac{2\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3}}{2\sqrt{7}}$, and $PQ = AP - AQ = \frac{3\sqrt{3}}{2\sqrt{7}}$. Since $\triangle BQP$ is a right triangle with hypotenuse BP, we see that $PQ^2 + BQ^2 = BP^2$. Dividing by BP^2 yields $\frac{(3\sqrt{3})^2}{(2\sqrt{7})^2} + \frac{BQ^2}{BP^2} = 1$, simplifying to $\frac{BQ}{BP} = \frac{1}{2\sqrt{7}}$. This means $\triangle AQB$ has $\frac{AQ}{BQ} = \sqrt{3}$, and so $\triangle AQB$ is a 30-60-90 triangle. By similarity, $\triangle APC$ is also a 30-60-90 right triangle.

From here, we can set $AO = 1$ (since we only are given ratios of lengths, we can pick AO to make things easier). Then $\triangle APC$ has $AP = 4\sqrt{3}$, $CP = 4$, $AC = 8$. The triangle has area $\frac{1}{2} \cdot AP \cdot PC = 8\sqrt{3}$. Then, letting $\overline{PD}$ be the altitude from P to $\overline{AC}$, we have that $\frac{1}{2} \cdot AC \cdot PD = 8\sqrt{3}$, and so $PD = 2\sqrt{3}$. By the similarity of $\triangle CDP$ to $\triangle CPA$ (they are both 30-60-90 right triangles), we have that $DP = 2$ and so $OD = 5$, and therefore $OP = \sqrt{OD^2 + DP^2} = \sqrt{25 + 12} = \sqrt{37}$, and since $AO = 1$ we have that $\frac{OP}{AO} = \boxed{\sqrt{37}}$.

Solution 2: Let K be the center of ω_1. We assign coordinates (setting the radius of ω_2, AO, to 1): $A = (-4, 0), O = (-3, 0), B = (-2, 0), K = (0, 0), P = (x, y)$. Then we have the two equations $x^2 + y^2 = 16$, $\frac{(x+4)^2+y^2}{(x+2)^2+y^2} = \frac{12}{7}$ by the conditions that P is on ω_1 and $\frac{AP}{BP} = \frac{2\sqrt{3}}{\sqrt{7}}$. We can then substitute $x^2 + y^2 = 16$ on the top and bottom of both of these equations to get $\frac{16+(8x+16)}{16+(4x+4)} = \frac{12}{7}$, which we can solve for x quickly:

$$\begin{aligned}\frac{16 + (8x + 16)}{16 + (4x + 4)} &= \frac{12}{7}\\ \frac{8x + 32}{4x + 20} &= \frac{12}{7}\\ 56x + 224 &= 48x + 240\\ x &= 2\end{aligned}$$

Plugging this into $x^2 + y^2 = 16$ gives $y = 2\sqrt{3}$, and then we can compute $OP^2 = (2 - (-3))^2 + (2\sqrt{3} - 0)^2 = 37$, and $OP = \sqrt{37}$. Thus $\frac{OP}{AO} = \frac{\sqrt{37}}{1} = \boxed{\sqrt{37}}$.

19. Consider an n-digit number $\underline{d_1}\,\underline{d_2}\,\underline{d_3} \ldots \underline{d_n}$ such that:

- there are no leading zeroes,
- the number formed from the first k digits $\underline{d_1} \ldots \underline{d_k}$ is divisible by k (for all $1 \le k \le n$), and
- all of the digits are either 0, 2, or 4.

If the number ends in the digits $\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$, what's the minimum value the number can be?

Answer: 240402402024

Solution: We make use of a lot of divisibility rules: namely those for 3,4,5, 8, and 11. With the properties listed, if the number is broken up into chunks of 3 digits starting from the left, those chunks must each be divisible by 3. In fact, the sum of these digits in these chunks must also be divisible by 3, by the standard divisibility rule for 3. Furthermore, every 5th digit starting from the left must be a 0, to ensure divisiblity by 5. We see that 2024 doesn't work as 202 is not divisible by 3. Now we need a number with at least 5 digits, so the 5th digit from the left must be 0. This pushes us to needing at least 7 digits $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$ as we need to make the 0 in 2024 the 5th digit. However, this doesn't work either as $\underline{d_4}\,\underline{d_5}\,\underline{d_6} = 202$ is not divisible by 3 as required. So the next possibility would be 9 digits: $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{d_4}\,\underline{d_5}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$. This doesn't work because $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{d_4}\,\underline{d_5}\,\underline{2}\,\underline{0}\,\underline{2}$ is not divisible by 8. We can tell this because it ends in 02, which means it is not divisible by 4 or 8.

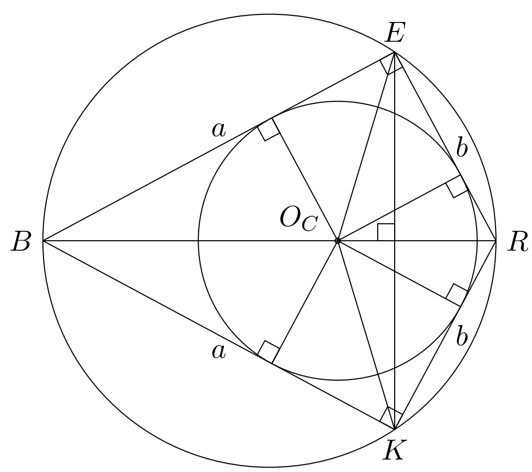
So, the first length that works is a 12 digit number $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{d_4}\,\underline{d_5}\,\underline{d_6}\,\underline{d_7}\,\underline{d_8}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$. We know that $d_5 = 0$ from our earlier reasoning. Working backwards, we see that we need $\underline{d_7}\,\underline{d_8}\,\underline{d_9} = \underline{d_7}\,\underline{d_8}\,\underline{2}$ to be divisible by 3 and $\underline{d_6 d_7 d_8}$ to be divisible by 8. In fact, we need $\underline{d_7}\,\underline{d_8}$ divisible by 8 since d_6 contributes $100d_6$ to the value of the first 8 digits, which is a multiple of 8 as d_6 is always a multiple of 2. We note that this implies $d_7+d_8 = 4$, and then we need $\underline{d_7}\,\underline{d_8} = 04, 22, 40$. However, of these only 40 is divisible by 8 and so $\underline{d_7}\,\underline{d_8} = 40$. This leaves us with $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{d_4}\,\underline{0}\,\underline{d_6}\,\underline{4}\,\underline{0}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$.

Now, we know that $d_4 + d_6$ is divisible by 3, and that $d_4 = 0, 4$ in order to satisfy the constraint on 4 digits. So $d_6 = 0, 2$. However, if $d_4 = d_6 = 0$, then our 11 constraint can't be satisfied, because we'll require $d_2 + 0 + 0 + 0 + 0 + 4 = d_1 + d_3 + 0 + 4 + 2 + 2$, which means $d_2 = d_1 + d_3 + 4$. This would need $d_2 = 4, d_1 = d_3 = 0$, but the leading digit can't be 0 so this is impossible. So, $d_4 = 4$ and $d_6 = 2$, giving $\underline{d_1}\,\underline{d_2}\,\underline{d_3}\,\underline{4}\,\underline{0}\,\underline{2}\,\underline{4}\,\underline{0}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{4}$. Finally, we need $d_1 + d_3 + 2 = d_2$ by applying the 11 divisibility constraint (the alternating sum of digits must be a multiple of 11), and since $d_1 \neq 0$ we have $d_1 = 2, d_2 = 4, d_3 = 0$ giving a final answer of $\boxed{240402402024}$.

20. Let U and C be two circles, and kite $BERK$ have vertices that lie on U and sides that are tangent to C. Given that the diagonals of the kite measure 5 and 6, find the ratio of the area of U to the area of C.

Answer: $\mathbf{\frac{66}{25}}$

Solution:



Note: there are multiple possible diagrams of $BERK$, but the final ratio of areas remains the same.

Let $BERK$ have side lengths a, a, b, b for some a and b. The kite's longer diagonal is a diameter of U due to the symmetrical nature of a kite, so U has radius $r_U = 3$ and area $A_U = 9\pi$. Moreover, splitting the kite in half shows that triangles $\triangle BER, \triangle BKR$ with side lengths a, b, 6 are right, so it follows from the Pythagorean theorem that

$$a^2 + b^2 = 6^2 = 36.$$

Then, notice that $\triangle BER$ has base $BR = 6$ and height $\frac{EK}{2} = \frac{5}{2}$. Computing the area of $\triangle BER$ two ways gives

$$\frac{1}{2}ab = \frac{1}{2} \cdot \frac{5}{2} \cdot 6 = \frac{15}{2}$$

or $ab = 15$. Combining these two equations yields

$$(a+b)^2 = a^2 + 2ab + b^2 = 36 + 2(15) = 66.$$

Let O_C denote the center of incircle C, r_C denote its radius, and $[\cdot]$ denote area. Then

$$[BERK] = [BO_CE] + [EO_CR] + [RO_CK] + [KO_CB] = 2\left(\frac{ar_C}{2} + \frac{br_C}{2}\right) = (a+b)r_C.$$

Since $[BERK] = 2 \cdot \frac{1}{2}ab = 15$, we conclude that

$$A_C = r_C^2\pi = \left(\frac{15}{a+b}\right)^2 \pi = \frac{225}{66}\pi = \frac{75}{22}\pi,$$

and the ratio of areas is

$$\frac{A_U}{A_C} = \frac{9\pi}{\frac{75}{22}\pi} = \boxed{\frac{66}{25}}.$$

21. Clara has a pair of nine-sided fair dice, each of whose faces are labeled $1, 2, 3, \ldots, 9$. Justin also has a pair of nine-sided fair dice, and the faces of his dice have positive integer labels, but one of Justin's dice has the number 13. When Clara and Justin roll their dice, it turns out that for every number S, the probability that the sum of the results of Clara's dice is S is equal to the probability that the sum of the results of Justin's dice is S. What is the probability that when Justin's dice are rolled, the results of the dice are equal?

Answer: $\mathbf{\frac{5}{81}}$

Solution 1: We are given that the distributions of the sums of Clara's dice and Justin's dice are the same. Included in the 81 equally likely pairs of results are:

- One pair summing to 2.
- Two pairs summing to 3.
- Three pairs summing to 4.
- Two pairs summing to 17.
- One pair summing to 18.

Of Justin's dice, let die A have a face with the number 13 on it, and let die B have a largest face value of b. Note that $1 + 1 = 2$ is the only way to get a sum of 2 with positive integers, so there must be a 1 on each die. Additionally, the largest sum 18 must be the sum of the two largest values of each die, meaning the largest value on both dice must only appear once on that die. So, the greatest face on die B is $18 - b$, which is at least 13, so $b \leq 5$.

Now, we can use the following logic:

- Since $1 + 2 = 3$ and 1 is a face on die A, there are at most two 2's on die B.
- Since $1 + 3 = 4$ and 1 is a face on die A, there are at most three 3's on die B.
- So, there are at most $1 + 2 + 3 = 6$ faces on die B that are less than 4, which forces $b > 4$ (otherwise die B would have at most 7 total faces, since there could only be one 4 if $b = 4$). Therefore, $b = 5$.
- Since $5 + 13 = 18$ and 13 is a face on die A, there is at most one 5 on die B.

- Since $5 + 12 = 17$ and 13 is a die on die A, there is at most two 4's on die B.
- But now, die B has a total of 9 faces if and only if the maximum number of number labels is achieved for each of the five observations made above.

We conclude that the distribution of die B is $(1, 2, 2, 3, 3, 3, 4, 4, 5)$. Now, we can build the faces of die A by matching the distribution of Clara's dice, and we get that the distribution of die A is $(1, 4, 4, 7, 7, 7, 10, 10, 13)$. The two dice have the same results if they are both 1 or both 4, and the probability that this happens is $\frac{1}{9} \cdot \frac{1}{9} + \frac{2}{9} \cdot \frac{2}{9} = \boxed{\frac{5}{81}}$.

Solution 2: We use generating functions: the results of each of Clara's nine-sided dice can be represented as

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 = x(1 + x + x^2)(1 + x^3 + x^6).$$

So the generating function for the sum of Clara's dice is $x^2(1 + x + x^2)^2(1 + x^3 + x^6)^2$, which must also be the generating function for the sum of Justin's dice. To create Justin's dice, we first allocate an x to each die to obtain positive face values. In order for each of Justin's dice to have 9 outcomes, the sum of the coefficients of the generating function of each die must be 9, so we allocate 2 of the factors in $(1 + x + x^2)^2(1 + x^3 + x^6)^2$ to each die. To make sure one of the generating functions of Justin's dice has a degree of at least 13, we can separate $(1 + x + x^2)^2$ and $(1 + x^3 + x^6)^2$, so that the generating functions of Justin's dice are

$$x(1 + x + x^2)^2 = x + 2x^2 + 3x^3 + 2x^4 + x^5$$

and

$$x(1 + x^3 + x^6)^2 = x + 2x^4 + 3x^7 + 2x^{10} + x^{13}.$$

So, the distribution of the faces of Justin's dice are $(1, 2, 2, 3, 3, 3, 4, 4, 5)$ and $(1, 4, 4, 7, 7, 7, 10, 10, 13)$. As in the previous solution, the two dice have the same results if they are both 1 or both 4, and the probability that this happens is $\frac{1}{9} \cdot \frac{1}{9} + \frac{2}{9} \cdot \frac{2}{9} = \boxed{\frac{5}{81}}$.

Note: In Solution 2, we don't actually need to know that one of Justin's dice has the number 13: there is exactly one different pair of nine-sided dice with the same distribution! Pairs of dice with unconventional face values whose distributions of sums are the same as a regular pair of dice are known as *Sicherman dice*.

22. There exist nonzero real numbers B, M, and T that satisfy the equations:

$$2B + M + T - 2B^2 - 2BM - 2MT - 2BT = 0,$$
$$B + 2M + T - 3M^2 - 3BM - 3MT - 3BT = 0,$$
$$B + M + 2T - 4T^2 - 4BM - 4MT - 4BT = 0.$$

Compute $2B + 3M + 4T$.

Answer: 3

Solution: First, rewrite as:

$$2B + M + T = 2B^2 + 2BM + 2MT + 2BT,$$
$$B + 2M + T = 3M^2 + 3BM + 3MT + 3BT,$$
$$B + M + 2T = 4T^2 + 4BM + 4MT + 4BT.$$

Upon factoring:

$$\begin{aligned} 2B + M + T &= 2(B+M)(B+T), \\ B + 2M + T &= 3(B+M)(M+T), \\ B + M + 2T &= 4(B+T)(M+T). \end{aligned}$$

Let $x = B + M$, $y = B + T$, and $z = M + T$. Then we have:

$$\begin{aligned} x + y &= 2xy, \\ x + z &= 3xz, \\ y + z &= 4yz. \end{aligned}$$

Dividing the first equation by xy, the second equation by xz, and the third equation by yz, we then have:

$$\begin{aligned} \frac{1}{y} + \frac{1}{x} &= 2, \\ \frac{1}{z} + \frac{1}{x} &= 3, \\ \frac{1}{z} + \frac{1}{y} &= 4. \end{aligned}$$

Solving this system gives $(x, y, z) = (2, \frac{2}{3}, \frac{2}{5})$. Now to solve for B, M, and T:

$$\begin{aligned} B + M &= 2, \\ B + T &= \frac{2}{3}, \\ M + T &= \frac{2}{5}. \end{aligned}$$

Solving this system gives $(B, M, T) = (\frac{17}{15}, \frac{13}{15}, -\frac{7}{15})$. Therefore, our answer is $2B + 3M + 4T = 2(\frac{17}{15}) + 3(\frac{13}{15}) + 4(-\frac{7}{15}) = \boxed{3}$.

Alternatively, once finding x, y, z you may obtain that $2B + 3M + 4T = \frac{3}{2}(x + y + z) - x + z = \frac{3}{2}(2 + \frac{2}{3} + \frac{2}{5}) - 2 + \frac{2}{5} = \boxed{3}$.

23. Find the greatest multiple of 43 whose base 6 representation is a permutation of the digits $1, 2, 3, 4$, and 5. (Express your answer in base 10).

Answer: 6020

Solution: Let N be a positive multiple of 43, and let its base-6 representation be $abcde_6$. Note that $abcde_6 \equiv a + b + c + d + e \pmod 5$. Since a, b, c, d, e are $1, 2, 3, 4, 5$ in some order, their sum is 15, which is divisible by $5 = 6 - 1$. Therefore, N is divisible by 5, so N is divisible by $43 \cdot 5 = 215 = 6^3 - 1$. Since $6^3 \equiv 6^0 \pmod{215}$, we have $0 \equiv abcde_6 \equiv ab_6 + cde_6 \pmod{215}$. But $0 < ab_6 + cde_6 < 2 \cdot 555_6$, so we must have $ab_6 + cde_6 = 555_6$.

Since 5 is the largest digit in base 6, no carrying occurs in the addition $0ab_6 + cde_6$. So, the pairs of digits in each place for the addition must be $(0, 5)$, $(1, 4)$, and $(2, 3)$. To maximize N, we set $0ab_6 = 043_6$, which forces $cde_6 = 512_6$. So, $N = 43512_6 = \boxed{6020}$.

24. Compute the number of positive integer triples (B, M, T) satisfying $B, M, T < 24$ and

$$BM + MT + BT = (B + M + T)\sqrt[3]{BMT}.$$

Answer: 89

Solution: Let B, M, T be the roots of a polynomial

$$x^3 - px^2 + qx - r = (x - B)(x - M)(x - T).$$

We have $p = B + M + T$, $q = BM + MT + BT$, and $r = BMT$ by Vieta's formulae. Thus, we see that the equation can be rewritten as $q = p\sqrt[3]{r}$, which is equivalent to $q^3 = p^3 r$, or $r = \frac{q^3}{p^3}$. Now, notice that $x = \frac{q}{p}$ is a root of $x^3 - px^2 + qx - r$ because

$$\left(\frac{q}{p}\right)^3 - p\left(\frac{q}{p}\right)^2 + q\left(\frac{q}{p}\right) - r = 0.$$

Therefore,

$$\left(\frac{q}{p} - B\right)\left(\frac{q}{p} - M\right)\left(\frac{q}{p} - T\right) = 0$$
$$(q - pB)(q - pM)(q - pT) = 0$$
$$(MT - B^2)(BT - M^2)(BM - T^2) = 0,$$

after plugging in $p = B + M + T$, $q = BM + MT + BT$, and $r = BMT$.

Thus, we have three cases: $B^2 = MT$, $M^2 = BT$, or $T^2 = BM$. This means B, M, T forms a geometric sequence in some order.

For $B, M, T < 24$, we have the following triples which can be rearranged in any way: $(1, 2, 4)$, $(1, 3, 9)$, $(1, 4, 16)$, $(2, 4, 8)$, $(2, 6, 18)$, $(3, 6, 12)$, $(4, 6, 9)$, $(4, 8, 16)$, $(5, 10, 20)$, $(8, 12, 18)$, and $(9, 12, 16)$.

Since there are 11 triples that can be arranged in any way, there are actually $11 \cdot 3! = 66$ total possibilities when B, M, T are distinct. However, we should not forget the trivial case where $B = M = T$. Thus, we have another 23 cases to add.

Therefore, the total number of triples (B, M, T) is $66 + 23 = \boxed{89}$.

25. For an arbitrary positive integer n, we define $f(n)$ to be the number of ordered 5-tuples of positive integers, $(a_1, a_2, a_3, a_4, a_5)$, such that $a_1 a_2 a_3 a_4 a_5 \mid n$. Compute the sum of all n for which $f(n)/n$ is maximized.

Answer: 2160

Solution: We compute $f(n)$ by considering each prime factor of n separately. Suppose the prime factorization of n is $\prod_{i=1}^{k} p_i^{e_i}$. Then, the divisibility condition is equivalent to the following: for each prime p_i dividing n, if the powers of p_i in the prime factorizations of $a_1, \ldots, a_5$ are $x_1, \ldots, x_5$, respectively, then we must have $x_1 + \ldots + x_5 \leq e_i$.

So for each prime p_i, we must count the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 \leq e_i$ where the x_i are nonnegative integers. This is equivalent to ordering e_i balls and 5 dividers, which split the e_i balls into 6 groups: the first five groups correspond to the x_i, and the last group is for the left over since the sum of the x_i can also be less than e_i. Hence there are $\binom{e_i+5}{5}$ ways to choose the exponents x_i for this prime.

This process can be repeated for each prime $p_i \mid n$, and to compute the total number of ways to choose the a_i we simply multiply the number of ways to choose the exponents for each prime, so $f(n) = \prod_{i=1}^{k} \binom{e_i+5}{5}$. Thus, $f(n)/n$ is

$$f(n)/n = \prod_{i=1}^{k} \frac{\binom{e_i+5}{5}}{p_i^{e_i}}.$$

To maximize this, we again consider one prime at a time. For each prime p, note that increasing the exponent of p in n from e to $e+1$ will multiply $f(n)/n$ by the quantity

$$\frac{\binom{e+6}{5}}{p\binom{e+5}{5}} = \frac{1}{p} \cdot \frac{e+6}{e+1}.$$

Note that this decreases as e increases, and for each p, we should only increase its exponent as long as this expression is greater than or equal to 1 (if it equals 1, then increasing the exponent would not change the value of $f(n)/n$). Hence for $p = 2$, the optimal exponent is either 4 or 5 (since $\frac{1}{6} \cdot \frac{4+6}{4+1} = 1$); for $p = 3$, we take the exponent to be 2 since $\frac{1}{3} \cdot \frac{1+6}{1+1} > 1$ but $\frac{1}{3} \cdot \frac{2+6}{2+1} < 1$; similarly, for $p = 5$, we can find the optimal exponent to be 1, and for $p \geq 7$, the expression $\frac{e+6}{p(e+1)}$ is always less than 1 so we ignore these primes.

Hence the values of n which maximize $f(n)/n$ are $2^4 \cdot 3^2 \cdot 5 = 720$ and $2^5 \cdot 3^2 \cdot 5 = 1440$, which sum to $\boxed{2160}$.