1. Find the third-largest three-digit multiple of three that is a palindrome. (Recall that a palindrome is a number that reads the same forward and backward, such as 444 or 838, but not 227.)

   Answer: 939

   Solution: In order for a three-digit number to be a palindrome, its units and hundreds digit must be equal, and for a number to be a multiple of three its digits must sum to a multiple of three. So, a number of the form $\underline{9}\underline{A}\underline{9}$ must have that $9 + A + 9 = 18 + A$ is a multiple of three, which means that A is a multiple of three. Therefore, the largest three-digit multiple of three that is a palindrome is 999, the second largest is 969, and the third largest is $\boxed{939}$.

2. The 35-step staircase of Sather Tower is being renovated. Each step will be painted a single color such that the stairs repeat color every 5 steps. There are 14 available stair colors, including blue and gold. Each color may only cover up to 10 steps, but both blue and gold must be used. With these restrictions, in how many different ways can the stairs be colored?

   Answer: 26400

   Solution: Since the pattern of 5 steps repeats 7 times, the number of stairs of any given color must be a multiple of 7. But there is only enough paint for up to 10 stairs of one color, so each color can be used at most once in the pattern. There are 5×4 ways to choose which stairs in the pattern receive the blue and gold colors. Then, for the remaining 3 stairs in the pattern, there are $\frac{12!}{9!} = 12 \times 11 \times 10$ ways to color them. So, the total number of ways is

$$5 \times 4 \times 12 \times 11 \times 10 = 20 \times 132 \times 10 = \boxed{26400}.$$

3. Find the number of positive integers, n, such that $\dfrac{20+n}{24-n}$ is an integer.

   Answer: 11

   Solution: We are given that $\dfrac{20+n}{24-n}$ is an integer. We can simplify the numerator as follows:

$$\frac{20+n}{24-n} = \frac{20+n}{24-n} + \frac{24-n}{24-n} - 1 = \frac{20+n+24-n}{24-n} - 1 = \frac{44}{24-n} - 1.$$

   So, we want to determine when $\dfrac{44}{24-n}$ is an integer. This occurs whenever $24 - n$ is a factor of 44. Since n is positive, $24 - n$ is less than 24, so the possible values of $24 - n$ are the positive and negative divisors of 44 that are less than 24. These include $\pm 1, \pm 2, \pm 4, \pm 11, \pm 22, -44$. We conclude that there are $\boxed{11}$ possible values of n.

4. Eight players are seated around a circular table. Each player is assigned to either Team Green or Team Yellow so that each team has at least one player. In how many ways can the players be assigned to the teams such that each player is on the same team as at least one player adjacent to them?

   Answer: 44

   Solution: Since not everyone can be on the same team, there are an even number of contiguous groups of players on the same team, alternating between Green and Yellow. There are either four groups of two players on the same team, or there are two contiguous groups on the opposing teams. In the former case, there are 4 ways (not 8, since this configuration has 180° rotational symmetry). In the latter case divides between Green and Yellow may be 2 vs. 6, 3 vs. 5, 4 vs. 4, 5 vs. 3, or 6 vs. 2, and each of these cases has 8 ways for each rotation around the table. There are a total of $4 + 8 \cdot 5 = \boxed{44}$ ways.

5. Clara has a pair of nine-sided fair dice, each of whose faces are labeled $1, 2, 3, \ldots, 9$. Justin also has a pair of nine-sided fair dice, and the faces of his dice have positive integer labels, but one of Justin's dice has the number 13. When Clara and Justin roll their dice, it turns out that for every number S, the probability that the sum of the results of Clara's dice is S is equal to the probability that the sum of the results of Justin's dice is S. What is the probability that when Justin's dice are rolled, the results of the dice are equal?

Answer: $\mathbf{\frac{5}{81}}$

Solution 1: We are given that the distributions of the sums of Clara's dice and Justin's dice are the same. Included in the 81 equally likely pairs of results are:

- One pair summing to 2.
- Two pairs summing to 3.
- Three pairs summing to 4.
- Two pairs summing to 17.
- One pair summing to 18.

Of Justin's dice, let die A have a face with the number 13 on it, and let die B have a largest face value of b. Note that $1 + 1 = 2$ is the only way to get a sum of 2 with positive integers, so there must be a 1 on each die. Additionally, the largest sum 18 must be the sum of the two largest values of each die, meaning the largest value on both dice must only appear once on that die. So, the greatest face on die B is $18 - b$, which is at least 13, so $b \leq 5$.

Now, we can use the following logic:

- Since $1 + 2 = 3$ and 1 is a face on die A, there are at most two 2's on die B.
- Since $1 + 3 = 4$ and 1 is a face on die A, there are at most three 3's on die B.
- So, there are at most $1 + 2 + 3 = 6$ faces on die B that are less than 4, which forces $b > 4$ (otherwise die B would have at most 7 total faces, since there could only be one 4 if $b = 4$). Therefore, $b = 5$.
- Since $5 + 13 = 18$ and 13 is a face on die A, there is at most one 5 on die B.
- Since $5 + 12 = 17$ and 13 is a die on die A, there is at most two 4's on die B.
- But now, die B has a total of 9 faces if and only if the maximum number of number labels is achieved for each of the five observations made above.

We conclude that the distribution of die B is $(1, 2, 2, 3, 3, 3, 4, 4, 5)$. Now, we can build the faces of die A by matching the distribution of Clara's dice, and we get that the distribution of die A is $(1, 4, 4, 7, 7, 7, 10, 10, 13)$. The two dice have the same results if they are both 1 or both 4, and the probability that this happens is $\frac{1}{9} \cdot \frac{1}{9} + \frac{2}{9} \cdot \frac{2}{9} = \boxed{\frac{5}{81}}$.

Solution 2: We use generating functions: the results of each of Clara's nine-sided dice can be represented as

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 = x(1 + x + x^2)(1 + x^3 + x^6).$$

So the generating function for the sum of Clara's dice is $x^2(1 + x + x^2)^2(1 + x^3 + x^6)^2$, which must also be the generating function for the sum of Justin's dice. To create Justin's dice, we first allocate an x to each die to obtain positive face values. In order for each of Justin's dice to have 9 outcomes, the sum of the coefficients of the generating function of each die must be 9, so we allocate 2 of the factors in $(1+x+x^2)^2(1+x^3+x^6)^2$ to each die. To make sure one of the generating functions of Justin's dice has a degree of at least 13, we can separate $(1+x+x^2)^2$ and $(1+x^3+x^6)^2$, so that the generating functions of Justin's dice are

$$x(1+x+x^2)^2 = x + 2x^2 + 3x^3 + 2x^4 + x^5$$

and

$$x(1+x^3+x^6)^2 = x + 2x^4 + 3x^7 + 2x^{10} + x^{13}.$$

So, the distribution of the faces of Justin's dice are $(1, 2, 2, 3, 3, 3, 4, 4, 5)$ and $(1, 4, 4, 7, 7, 7, 10, 10, 13)$. As in the previous solution, the two dice have the same results if they are both 1 or both 4, and the probability that this happens is $\frac{1}{9} \cdot \frac{1}{9} + \frac{2}{9} \cdot \frac{2}{9} = \boxed{\frac{5}{81}}$.

Note: In Solution 2, we don't actually need to know that one of Justin's dice has the number 13: there is exactly one different pair of nine-sided dice with the same distribution! Pairs of dice with unconventional face values whose distributions of sums are the same as a regular pair of dice are known as *Sicherman dice*.

6. Find the greatest multiple of 43 whose base 6 representation is a permutation of the digits $1, 2, 3, 4$, and 5. (Express your answer in base 10).

Answer: 6020

Solution: Let N be a positive multiple of 43, and let its base-6 representation be $abcde_6$. Note that $abcde_6 \equiv a+b+c+d+e \pmod 5$. Since a, b, c, d, e are $1, 2, 3, 4, 5$ in some order, their sum is 15, which is divisible by $5 = 6-1$. Therefore, N is divisible by 5, so N is divisible by $43 \cdot 5 = 215 = 6^3 - 1$. Since $6^3 \equiv 6^0 \pmod{215}$, we have $0 \equiv abcde_6 \equiv ab_6 + cde_6 \pmod{215}$. But $0 < ab_6 + cde_6 < 2 \cdot 555_6$, so we must have $ab_6 + cde_6 = 555_6$.

Since 5 is the largest digit in base 6, no carrying occurs in the addition $0ab_6 + cde_6$. So, the pairs of digits in each place for the addition must be $(0, 5)$, $(1, 4)$, and $(2, 3)$. To maximize N, we set $0ab_6 = 043_6$, which forces $cde_6 = 512_6$. So, $N = 43512_6 = \boxed{6020}$.

7. Gigi randomly rearranges four G's and seven I's to form an eleven-letter string. What is the probability that there is a group of four consecutive letters that form "$GIGI$," her name?

Answer: $\frac{2}{5}$

Solution: There are $\binom{11}{4} = 330$ equally likely permutations of four G's and seven I's. Note that the four G's divide the seven I's into five groups; let a_1, a_2, a_3, a_4, and a_5 be the number of I's in each group from left to right, so that $a_1 + a_2 + a_3 + a_4 + a_5 = 7$. To find the number of eleven-letter strings that contain "$GIGI$", we rephrase this condition as follows: $a_2 = 1$ and $a_3 \geq 1$, or $a_3 = 1$ and $a_4 \geq 1$, or $a_4 = 1$ and $a_5 \geq 1$. This is because $a_2 = 1$ or $a_3 = 1$ creates a "GIG" sequence, and the next group should contain at least one I. Let the sets of 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ summing to 7 and satisfying each respective condition be A, B, and C.

We want to find $|A \cap B \cap C|$, so we use Principle of Inclusion-Exclusion, using the nonnegative integer version of sticks-and-stones to enumerate each intersection of sets:

- $|A|$, and similarly for $|B|$ and $|C|$: $a_2 = 1, a_3 \geq 1 \longrightarrow a_1 + (a_3 - 1) + a_4 + a_5 = 7 - 2 = 5$, in $\binom{5+4-1}{4-1} = \binom{8}{3} = 56$ ways
- $|A \cap B|$, and similarly for $|B \cap C|$: $a_2 = 1, a_3 = 1, a_4 \geq 1 \longrightarrow a_1 + (a_4 - 1) + a_5 = 7 - 3 = 4$, in $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ ways

- $|A \cap C|$: $a_2 = 1, a_3 \geq 1, a_4 = 1, a_5 \geq 1 \longrightarrow a_1 + (a_3 - 1) + (a_5 - 1) = 7 - 4 = 3$, in $\binom{3+3-1}{3-1} = \binom{5}{2} = 10$ ways
- $|A \cap B \cap C|$: $a_2 = 1, a_3 = 1, a_4 = 1, a_5 \geq 1 \longrightarrow a_1 + (a_5 - 1) = 7 - 4 = 3$, in $\binom{3+2-1}{2-1} = \binom{4}{1} = 4$ ways

The total number of eleven-letter strings that contain "$GIGI$" is

$$\begin{aligned} &|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 56 + 56 + 56 - 15 - 10 - 15 + 4 = 132. \end{aligned}$$

So the probability is $\dfrac{132}{330} = \boxed{\dfrac{2}{5}}$.

8. For an arbitrary positive integer n, we define $f(n)$ to be the number of ordered 5-tuples of positive integers, $(a_1, a_2, a_3, a_4, a_5)$, such that $a_1 a_2 a_3 a_4 a_5 \mid n$. Compute the sum of all n for which $f(n)/n$ is maximized.

Answer: 2160

Solution: We compute $f(n)$ by considering each prime factor of n separately. Suppose the prime factorization of n is $\prod_{i=1}^{k} p_i^{e_i}$. Then, the divisibility condition is equivalent to the following: for each prime p_i dividing n, if the powers of p_i in the prime factorizations of $a_1, \ldots, a_5$ are $x_1, \ldots, x_5$, respectively, then we must have $x_1 + \ldots + x_5 \leq e_i$.

So for each prime p_i, we must count the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 \leq e_i$ where the x_i are nonnegative integers. This is equivalent to ordering e_i balls and 5 dividers, which split the e_i balls into 6 groups: the first five groups correspond to the x_i, and the last group is for the left over since the sum of the x_i can also be less than e_i. Hence there are $\binom{e_i+5}{5}$ ways to choose the exponents x_i for this prime.

This process can be repeated for each prime $p_i \mid n$, and to compute the total number of ways to choose the a_i we simply multiply the number of ways to choose the exponents for each prime, so $f(n) = \prod_{i=1}^{k} \binom{e_i+5}{5}$. Thus, $f(n)/n$ is

$$f(n)/n = \prod_{i=1}^{k} \frac{\binom{e_i+5}{5}}{p_i^{e_i}}.$$

To maximize this, we again consider one prime at a time. For each prime p, note that increasing the exponent of p in n from e to $e+1$ will multiply $f(n)/n$ by the quantity

$$\frac{\binom{e+6}{5}}{p\binom{e+5}{5}} = \frac{1}{p} \cdot \frac{e+6}{e+1}.$$

Note that this decreases as e increases, and for each p, we should only increase its exponent as long as this expression is greater than or equal to 1 (if it equals 1, then increasing the exponent would not change the value of $f(n)/n$). Hence for $p = 2$, the optimal exponent is either 4 or 5 (since $\frac{1}{6} \cdot \frac{4+6}{4+1} = 1$); for $p = 3$, we take the exponent to be 2 since $\frac{1}{3} \cdot \frac{1+6}{1+1} > 1$ but $\frac{1}{3} \cdot \frac{2+6}{2+1} < 1$; similarly, for $p = 5$, we can find the optimal exponent to be 1, and for $p \geq 7$, the expression $\frac{e+6}{p(e+1)}$ is always less than 1 so we ignore these primes.

Hence the values of n which maximize $f(n)/n$ are $2^4 \cdot 3^2 \cdot 5 = 720$ and $2^5 \cdot 3^2 \cdot 5 = 1440$, which sum to $\boxed{2160}$.

9. Compute the remainder when

$$\left(4^{1^2} + 4^{2^2} + 4^{3^2} + \cdots + 4^{82^2}\right)^2$$

is divided by 83.

Answer: 81

Solution: For brevity, let $q = 41$ and $p = 2q + 1 = 83$ be primes, and set $a = 4$ so that we are interested in evaluating

$$S = \left(\sum_{k=1}^{p-1} a^{k^2}\right)^2 = \sum_{1 \le k, \ell \le 2q} a^{k^2+\ell^2}$$

modulo p.

The main point is that the exponent $k^2 + \ell^2$ is only considered $\pmod{2q}$, so the main part of the computation is determining the number of solutions to $k^2 + \ell^2 \equiv d \pmod{2q}$ for some $d \in \{1, 2, \ldots, 2q\}$ where $k, \ell \in \{1, 2, \ldots, 2q\}$. By the Chinese remainder theorem, it suffices to separate the problem into $\pmod 2$ and $\pmod q$ variants of this problem.

For the $\pmod 2$ problem, we note that the values $0^2 + 0^2$ and $0^2 + 1^2$ and $1^2 + 0^2$ and $1^2 + 1^2$ hits $0 \pmod 2$ twice and $1 \pmod 2$ twice. For the $\pmod q$ problem, we have two cases.

- Suppose $d \equiv 0 \pmod q$. Then we are counting the number of pairs $(k, \ell) \in \{1, 2, \ldots, q\}^2$ such that $k^2 + \ell^2 \equiv 0 \pmod q$. However, $q \equiv 1 \pmod 4$, so -1 is a (nonzero) square $\pmod q$, so it suffices to count pairs (k, ℓ) such that $k^2 - \ell^2 \equiv 0 \pmod q$, which is equivalent to $k \equiv \pm\ell \pmod q$. For each nonzero $k \pmod q$, there are two values of ℓ; otherwise, $k \equiv \ell \equiv 0 \pmod q$ has one solution. This totals to $2q - 1$ pairs (k, ℓ).
- Suppose $d \not\equiv 0 \pmod q$. Then we are counting the number of pairs $(k, \ell) \in \{1, 2, \ldots, q\}^2$ such that $k^2 + \ell^2 \equiv d \pmod q$. As before, we see that -1 is a nonzero square $\pmod q$, so it suffices to count pairs (k, ℓ) such that

$$(k + \ell)(k - \ell) = k^2 - \ell^2 \equiv d \pmod q.$$

  Doing a change of variables with $x = k + \ell$ and $y = k - \ell$, it suffices to count pairs (x, y) such that $xy \equiv d \pmod q$. However, this is equivalent to $x \equiv ay^{-1} \pmod q$, so we have $q - 1$ pairs (x, y) in this case.

Combining the above work with the Chinese remainder theorem, we see

$$\#\left\{(k, \ell) \in \{1, 2, \ldots, 2q\}^2 : k^2 + \ell^2 \equiv d \pmod{2q}\right\} = \begin{cases} 2(2q-1) & \text{if } d \equiv 0 \pmod q, \\ 2(q-1) & \text{if } d \not\equiv 0 \pmod q. \end{cases}$$

It follows that

$$S \equiv 2(q-1)\sum_{d=1}^{2q} a^d + 2q\left(a^0 + a^q\right).$$

Quickly, note that $a \not\equiv 1 \pmod p$ allows us to write

$$\sum_{d=1}^{2q} a^d \equiv \frac{a^{2q+1} - a}{a - 1} \equiv \frac{a^p - a}{a - 1} \equiv 0 \pmod p$$

using the formula of a partial sum of a geometric series followed by Fermat's little theorem. Finishing up, we see that

$$S \equiv (p-1)\left(4^0 + 4^{(p-1)/2}\right) \equiv -\left(1 + 2^{p-1}\right) \equiv -2 \pmod{p},$$

so our answer is $p - 2 = \boxed{81}$.

10. The positive integers 1 through 9 are placed in the 9 cells of a 3×3 grid. Then, for every pair of cells sharing a side, the sum of the numbers in that pair is recorded in a list. The most number of times any number occurs in the list is 4. In how many ways could numbers have been placed in the grid?

Answer: 20616

Solution: Let m be a mode of the list, so that $m = a_1 + a_2 = a_3 + a_4 = a_5 + a_6 = a_7 + a_8$ where $1 \le a_k \le 9$, and for $1 \le i \le 4$, $a_{2i-1} \ne a_{2i}$ and each unordered pair $\{a_{2i-1}, a_{2i}\}$ is distinct. Then each a_k is distinct: if WLOG $a_1 = a_3$, then $a_2 = m - a_1 = m - a_3 = a_4$, so $\{a_1, a_2\} = \{a_3, a_4\}$, which is a contradiction. Let a_9 be the remaining number from 1 to 9 distinct from the a_i's. Then $4m = \left(\sum_{i=1}^{9} a_i\right) - a_9 = 45 - a_9$, so $m = \dfrac{45 - a_9}{4}$. Since m is an integer, a_9 may only be 1, 5, or 9, which lead to unique pairings that sum to $m = 11$, 10, and 9, respectively.

We may place the numbers in the grid with the following process:

- First, we tile the grid with four 2×1 dominoes and one 1×1 square to establish the pairs of cells that sum to m.
- Then, we establish the value of a_9, and place it in the 1×1 square.
- Finally, we place the pairs of numbers summing to m in the dominoes.

To tile the grid, we may either place the 1×1 square in a corner or in the center; there is no tiling where the square is placed on a side cell. If the square is placed in one of the 4 corners, there are 4 ways to tile the rest of the grid with the dominoes. If the square is placed in the center, there are 2 ways to tile the rest of the grid with the dominoes. Thus, there are a total of $4 \cdot 4 + 2 = 18$ ways to tile the grid. There are 3 ways to set the value of a_9 to either 1, 5, or 9. Then, there are $4! = 24$ ways to assign the pairs of numbers to the dominoes, after which there are $2^4 = 16$ ways to choose which number in each pair goes in which cell of the pair's assigned domino. The total number of ways according to this method is $18 \cdot 3 \cdot 24 \cdot 16 = 20736$.

However, we overcount cases where the mode is not unique, and there are two possibilities for a_i. If a_i may be both 1 and 9 in a given grid, then the modes are 11 and 9, giving the chain of adjacent squares $1-8-3-6-5-4-7-2-9$. If a_i may be both 1 and 5 in a given grid, then the modes are 11 and 10, giving the chain of adjacent squares $1-9-2-8-3-7-4-6-5$. If a_i may be both 5 and 9 in a given grid, then the modes are 10 and 9, giving the chain of adjacent squares $9-1-8-2-7-3-6-4-5$. The chain could start from either a corner from which there are 8 different paths, or the center of the grid from which there are also 8 different paths. This gives $8 \cdot 4 + 8 = 40$ possible chains in all and thus $3 \cdot 40 = 120$ grids that are counted twice. Note that modes of 9, 10, and 11 cannot occur simultaneously: in that case, since there are exactly 12 adjacency pairs in the grid, the list of pair sums would be $9, 9, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11$. However, the middle cell of the grid yields four distinct sums (as it is adjacent to four distinct numbers), which is a contradiction. Thus, the total number of ways is $20736 - 120 = \boxed{20616}$.