1. Find
$$\lim_{x\to 2} \frac{x^3-8}{x-2}.$$

Answer: 12

Solution: Using the difference-of-cubes factorization, $\frac{x^3-8}{x-2} = \frac{(x-2)(x^2+2x+4)}{x-2}$. This expression is equivalent to x^2+2x+4 everywhere except $x=2$, where there is a hole in its graph. The limit bypasses this discontinuity, so the limit is:
$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} x^2+2x+4 = 2^2+2\cdot 2+4 = \boxed{12}.$$

2. Given $f(x) = x^3-3x^2+3x-1$ and $f(x) = g(44-x)$, find $f'(20)+g'(24)$.

Answer: 0

Solution: From $f(x) = g(44-x)$, take the derivative and use the chain rule to get $f'(x) = -g'(44-x)$, so $f'(x)+g'(44-x) = 0$. Plugging in $x = 20$ gives us $f'(20)+g'(24) = \boxed{0}$.

3. Let $f(x) = \frac{3x-14}{x^2-6x+8}$. Compute $f'''(3)$.

Answer: −18

Solution: Using partial fraction decomposition, we find that $f(x) = \frac{4}{x-2} - \frac{1}{x-4}$. Taking the third derivative of this using power rule, $f'''(x) = \frac{-24}{(x-2)^4} + \frac{6}{(x-4)^4}$. Plugging in $x = 3$, $f'''(3) = \boxed{-18}$.

4. Find the positive, real value of k where $e^{kx} = 3\sqrt{x}$ has exactly 1 solution.

Answer: $\mathbf{\frac{9}{2e}}$

Solution: We will first find the x-value where the intersection takes place in terms of k. Observe that
$$\begin{aligned} e^{kx} &= 3\sqrt{x} \\ ke^{kx} &= \frac{3}{2\sqrt{x}} \end{aligned}$$
by taking derivatives on both sides. Substituting and solving for x gives that
$$\begin{aligned} k(3\sqrt{x}) &= \frac{3}{2\sqrt{x}} \\ 2kx &= 1 \\ x &= \frac{1}{2k}. \end{aligned}$$
Now, plugging in $x = \frac{1}{2k}$ will give us some nice cancelation and let us solve for k:
$$\begin{aligned} e^{kx} &= 3\sqrt{x} \\ e^{\frac{1}{2}} &= 3\sqrt{\frac{1}{2k}} \\ \sqrt{2ek} &= 3 \\ 2ek &= 9 \\ k &= \boxed{\frac{9}{2e}} \end{aligned}$$

5. For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n. Compute

$$\lim_{n\to\infty} \int_0^n \frac{x\lfloor x \rfloor}{n^3} \, \mathrm{d}x \, .$$

Answer: $\mathbf{\frac{1}{3}}$

Solution: We can rewrite the integral as a sum:

$$\begin{aligned}
\lim_{n\to\infty} \int_0^n \frac{x\lfloor x \rfloor}{n^3} \, \mathrm{d}x &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \int_k^{k+1} \frac{x\lfloor x \rfloor}{n^3} \, \mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k}{n^3} \int_k^{k+1} x \, \mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k}{2n^3} \left((k+1)^2 - k^2\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k(2k+1)}{2n^3} \\
&= \lim_{n\to\infty} \frac{n(n-1)}{4n^3} + \sum_{k=0}^{n-1} \frac{k^2}{n^2} \cdot \frac{1}{n}
\end{aligned}$$

The first summand goes to 0, and it's possible to use the summation formula to find the limit of the second summand. However, one can also recognize that the second summand is actually a Riemann sum, where $\frac{1}{n} = \Delta x$ and $f(k\Delta x) = (k\Delta x)^2$ from 0 to 1. This yields

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{k^2}{n^2} \cdot \frac{1}{n} = \int_0^1 x^2 \, \mathrm{d}x = \boxed{\frac{1}{3}}.$$

6. What is the smallest positive integer $n > 1$ such that

$$\int_1^n \sqrt{\sqrt{3+\sqrt{x}} - 2} \, \mathrm{d}x$$

is rational?

Answer: 36

Solution: Consider the u-substitution $u = \sqrt{\sqrt{3+\sqrt{x}} - 2}$. Then $((u^2+2)^2-3)^2 = x$ and $\mathrm{d}x = f(u)\mathrm{d}u$ for some polynomial f. After performing this u-subsitution, the integrand will become a polynomial in u with rational coefficients. So we need to choose n where the bounds of the new integral are rational: this means plugging in $x = n$ should give a rational u. In order for $\sqrt{\sqrt{3+\sqrt{n}} - 2}$ to be rational, we need n to be a square integer. We also need $3+\sqrt{n}$ to be a square integer. Since $n > 1$, the first possible value of $\sqrt{n}$ is 6 to make $3+\sqrt{n} = 9$. Therefore, $n = 36$ is a possibility, and we see that $n = 36$ gives $\sqrt{\sqrt{3+\sqrt{n}} - 2} = 1$ which is rational as needed. So, our answer is $\boxed{36}$.

7. Evaluate

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}} \, \mathrm{d}x\,.$$

Answer: $\mathbf{\frac{2}{3} - \frac{5}{3^{\frac{5}{2}}}}$

Solution:

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}} \, \mathrm{d}x &= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}(1+x)^2} \, \mathrm{d}x \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{2\sin(2\theta)}{\sin(2\theta)(1+\cos(2\theta))^2} \, \mathrm{d}\theta && (x = \cos(2\theta)) \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^4(\theta)} \, d\theta && (\cos(2\theta) = 2\cos^2(\theta) - 1) \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^4(\theta) \, \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1} \sec^2(\theta) \, \mathrm{d}u && (u = \tan(\theta)) \\
&= \frac{1}{2}\int_{\frac{\sqrt{3}}{3}}^{1} (u^2+1) \, \mathrm{d}u && (\tan^2(\theta) + 1 = \sec^2(\theta)) \\
&= \boxed{\frac{2}{3} - \frac{5}{3^{\frac{5}{2}}}}
\end{aligned}$$

8. Evaluate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{9n^2 - 12n + 2}{(3n)(3n-1)(3n-2)}.$$

Answer: $\mathbf{-\frac{\ln(2)}{3}}$

Solution: Partial fraction decomposition gives

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{-1}{3n-2}+\frac{1}{3n-1}+\frac{1}{3n}\right)\\
&= -1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\cdots\\
&= \int_0^1 -1\ dx+\int_0^1 x\ dx+\int_0^1 x^2\ dx+\int_0^1 x^3\ dx-\int_0^1 x^4\ dx+\cdots\\
&= \int_0^1 (-1+x+x^2)+(x^3-x^4-x^5)+(-x^6+x^7+x^8)+\cdots\ dx\\
&= \int_0^1 \frac{-1+x+x^2}{1+x^3}\ dx \quad \text{by geometric series formula}^{*}\\
&= \int_0^1 \frac{-(1-x+x^2)}{(1+x)(1-x+x^2)}+\frac{2x^2}{1+x^3}\ dx\\
&= \int_0^1 -\frac{1}{1+x}\ dx+\frac{2}{3}\int_1^2 \frac{1}{u}\ du\\
&= \boxed{-\frac{\ln(2)}{3}}
\end{aligned}$$

* We can use the geometric series formula because the common ratio of terms is $-x^3$ and $|x^3| < 1$ when strictly inside the bounds.

9. Given an infinite sequence of real numbers $x_0, x_1, x_2, \ldots$ where for $i \geq 0$,

$$x_{i+1} = \frac{4x_i^3 - 1}{6x_i^2 - 3},$$

there are exactly three possible real values, $a < b < c$, that $\lim_{n\to\infty} x_n$ may converge to, depending on x_0. A real number m satisfies the condition that, for all $x_0 < m$, the sequence converges to a. Find the maximum possible value of $m + a$.

Answer: $\mathbf{\frac{-1-\sqrt{2}-\sqrt{3}}{2}}$

Solution: Recurrences converging to a small set of numbers, regardless of *most* initial conditions, are reminiscent of the Newton-Raphson method, where

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We see that the sequence of (x_i) can only converge if the value of $\frac{f(x_i)}{f'(x_i)}$ goes to 0 in the limit, meaning the numerator $f(x_i)$ must go to 0 and (x_i) must therefore approach a root of $f(x)$.

Manipulating the given sequence to align with Newton-Raphson gives:

$$x_{i+1} = x_i - \frac{2x_i^3 - 3x_i + 1}{6x_i^2 - 3}$$

Thus $f(x_i) = 2x_i^3 - 3x_i + 1$. This is a cubic, so it has three roots which are our a, b, c. We can find a by computing the roots of $f(x) = (x-1)(2x^2+2x-1)$, which are $1, \frac{-2\pm\sqrt{12}}{4}$. Recall that a is the least root, which is $\frac{-2-\sqrt{12}}{4} = \frac{-1-\sqrt{3}}{2}$.

Now, consider the left-most stationary point (the least x such that $f'(x) = 0$). We know x_0 cannot be located on it as algebraically, the denominator of the sequence ($6x_i^2 - 3 = f'(x_i)$) will be zero. A geometric explanation is that this will produce a tangent line parallel to the x-axis, thus never continuing the sequence as the next element of the sequence should be the x-intercept of the tangent line.

Furthermore, f is increasing and concave down to the left of this stationary point. Geometrically, this means that any tangent line to f at $x_0 < m$ must intersect the x-axis to the left of the least root a, and so x_1 will be less than a. From there, we can see that $f(x_i)$ will always be negative and $f'(x_i)$ will always be positive for all $i \geq 1$, so the sequence $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ is increasing and bounded above by a, meaning it must converge. Thus m is the left-most stationary point, which is $\frac{-\sqrt{2}}{2}$ by setting $f'(x) = 0$.

Therefore, the final answer becomes $\boxed{\dfrac{-1-\sqrt{2}-\sqrt{3}}{2}}$.

10. Let a function $f(n)$ satisfy $f(1) = 0$, and for positive integers $n > 1$,

$$f(n) = \begin{cases} f(\frac{n}{2}) + \ln 2, & \text{if } n \text{ is even} \\ \frac{f(n-1)+f(n+1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Find the value of

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} |\ln k - f(k)|.$$

Answer: $\frac{3}{2}\ln 2 - 1$

Solution: First, we notice that $f(n)$ is approximately equal to $\ln n$. This is because $\ln 2n = \ln n + \ln 2$ for all $n > 0$ and $\ln x \approx \frac{\ln(x+1)+\ln(x-1)}{2}$ for all $x > 1$.

Our goal is to consider the sum as a Riemann sum in terms of a continuous function, which will allow us to evaluate it exactly as an integral when taking the limit. Therefore, we want a continuous function g defined on the interval $[1, \infty)$ that agrees with f on all positive integers. In order to create this function, we extend the restriction on f for even integers to all x: we require that $g(x) = g(\frac{x}{2}) + \ln 2$ for all inputs $x \geq 2$. Then, we only need to define $g(x)$ on the interval $[1, 2)$ and then it will be defined on $[1, \infty)$ by repeatedly dividing by 2 until x is within that range.

We extend the second property of f to require that the point $(x, g(x))$ be the midpoint of $(x - d, g(x - d))$ and $(x + d, g(x + d))$ for any x, d chosen so that $1 \leq x - d \leq x + d < 2$. We can see that any points constructed this way from the starting points $(1, 0), (2, \ln 2)$ must be on the segment between these two points, and these points can get arbitrarily close together. By continuity of g, g must be the segment from $(1, f(1))$ to $(2, f(2))$ (not including the endpoint at $x = 2$) or a line from $(1, 0)$ to $(2, \ln 2)$ on the interval $[1, 2)$, which has the equation $g(x) = \ln 2(x - 1)$ on that same interval. The general form of $g(x)$ becomes

$$g(x) = \begin{cases} \ln 2(\frac{x}{2^n} - 1) + n \ln 2, & x \in [2^n, 2^{n+1}) \end{cases}$$

for some integer $n \geq 0$, by applying our "halving property" that $g(x) = g(\frac{x}{2}) + \ln 2$. We claim that this definition of $g(x)$ has $g(n) = f(n)$ for positive integers n. We prove this by showing that $g(n)$ satisfies the same recurrence relation as $f(n)$: we know that $g(1) = f(1) = 0$, and then by definition of g we know that $g(n) = g(\frac{n}{2}) + \ln 2$ for even n since it applies for all $n \geq 2$. Then, if we consider $g(n)$ for n odd, if $n \in [2^k, 2^{k+1}]$ we see that $n \in [2^k + 1, 2^{k+1} - 1]$ since n is odd and so $n-1, n, n+1 \in [2^k, 2^{k+1}]$. Therefore, we have that

$$g\left(\frac{n}{2^k}\right) = \frac{1}{2}\left(g\left(\frac{n-1}{2^k}\right) + g\left(\frac{n+1}{2^k}\right)\right)$$

by applying our midpoint property. However, by adding $k\ln 2$ to both sides, we see that

$$g\left(\frac{n}{2^k}\right) + k\ln 2 = \frac{1}{2}\left(g\left(\frac{n-1}{2^k}\right) + g\left(\frac{n+1}{2^k}\right)\right) + k\ln 2$$
$$g\left(\frac{n}{2^k}\right) + k\ln 2 = \frac{1}{2}\left(g\left(\frac{n-1}{2^k}\right) + k\ln 2 + g\left(\frac{n+1}{2^k}\right) + k\ln 2\right)$$
$$g(n) = \frac{1}{2}\left(g(n-1) + g(n+1)\right)$$

by applying our halving property. So g satisfies the same recurrence relation f does on the positive integers, meaning that g must agree with f on the positive integers. In the desired sum, we can then replace $f(k)$ with $g(k)$, yielding our desired value S as

$$S = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} |\ln k - g(k)|$$
$$S = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} \left|\left(\ln\frac{k}{2^n} + n\ln 2\right) - \left(g\left(\frac{k}{2^n}\right) + n\ln 2\right)\right|$$
$$S = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} \left|\ln\frac{k}{2^n} - g\left(\frac{k}{2^n}\right)\right|$$

by taking advantage of log properties and the halving property of g. This is now the limit as n goes to infinity of a Riemann sum with 2^n equally sized subintervals on the interval $[1, 2]$, which means the limit evaluates to

$$\int_1^2 |\ln x - g(x)| \mathrm{d}x$$

Notice that ln is concave, so that $g(k) \leq \ln k$ as it is on a secant line of the function $\ln x$, and we may remove the absolute value. Then, we simply plug in $g(x)$ and evaluate:

$$\int_1^2 (\ln x - g(x)) \mathrm{d}x$$

$$\int_1^2 (\ln x - \ln 2(x-1)) \mathrm{d}x$$

$$\left[(x\ln x - x) - \left(x^2\frac{\ln 2}{2} - x\ln 2\right)\right]_1^2$$

$$(2\ln 2 - 2 - (2\ln 2 - 2\ln 2)) - \left(-1 - \left(\frac{1}{2}\ln 2 - \ln 2\right)\right) = \boxed{\frac{3}{2}\ln 2 - 1}$$