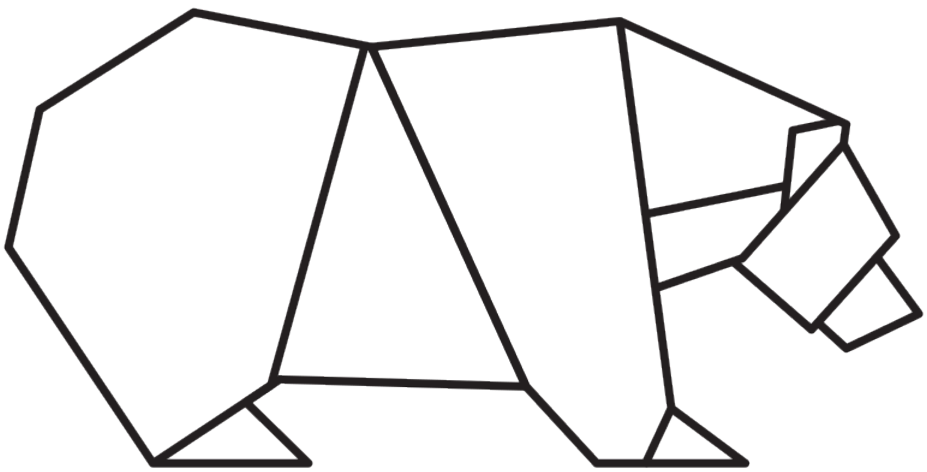
Berkeley Math Tournament 2024

Calculus Test



November 2, 2024

Time limit: 60 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators. Protractors, rulers, and compasses are permitted.

Answer format overview:

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2} + \frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π, e, $\sin(10°)$, and $\ln 2$ in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

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1. Find
$$\lim_{x\to 2} \frac{x^3-8}{x-2}.$$

2. Given $f(x) = x^3 - 3x^2 + 3x - 1$ and $f(x) = g(44 - x)$, find $f'(20) + g'(24)$.

3. Let $f(x) = \frac{3x-14}{x^2-6x+8}$. Compute $f'''(3)$.

4. Find the positive, real value of k where $e^{kx} = 3\sqrt{x}$ has exactly 1 solution.

5. For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n. Compute
$$\lim_{n\to\infty} \int_0^n \frac{x\lfloor x \rfloor}{n^3}\, \mathrm{d}x\,.$$

6. What is the smallest positive integer $n > 1$ such that
$$\int_1^n \sqrt{\sqrt{3+\sqrt{x}}-2}\, \mathrm{d}x$$
is rational?

7. Evaluate
$$\int_0^{\frac{1}{2}} \frac{1}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{5}{2}}}\, \mathrm{d}x\,.$$

8. Evaluate
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{9n^2-12n+2}{(3n)(3n-1)(3n-2)}.$$

9. Given an infinite sequence of real numbers $x_0, x_1, x_2, \ldots$ where for $i \geq 0$,
$$x_{i+1} = \frac{4x_i^3-1}{6x_i^2-3},$$
there are exactly three possible real values, $a < b < c$, that $\lim_{n\to\infty} x_n$ may converge to, depending on x_0. A real number m satisfies the condition that, for all $x_0 < m$, the sequence converges to a. Find the maximum possible value of $m + a$.

10. Let a function $f(n)$ satisfy $f(1) = 0$, and for positive integers $n > 1$,
$$f(n) = \begin{cases} f(\frac{n}{2}) + \ln 2, & \text{if } n \text{ is even} \\ \frac{f(n-1)+f(n+1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$
Find the value of
$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}} |\ln k - f(k)|.$$