1. At a certain point in time, Nikhil had 3 more apples than Brian. Theo then gave Nikhil 9 apples and took away 3 apples from Brian. Now, Nikhil has twice as many apples as Brian. Compute the number of apples that Nikhil and Brian now have in total.

   Answer: 45

   Solution: Suppose Nikhil starts with N apples and Brian with B apples. We then have

$$N = B + 3 \tag{1}$$

$$N + 9 = 2(B - 3) \tag{2}$$

   based on the given information. Substituting (1) into (2) gives $B + 12 = 2(B - 3)$, so $B + 12 = 2B - 6$, and $B = 18$. So, $N = 21$. Our final answer is then $(N + 9) + (B - 3) = 30 + 15 = \boxed{45}$.

2. Find the real number x satisfying

$$\frac{x^2 - 20}{x^2 + 20x + 4} = \frac{x^2 - 24}{x^2 + 24x + 4} = \frac{1}{2}.$$

   Answer: −2

   Solution: Working with $\frac{x^2-20}{x^2+20x+4} = \frac{1}{2}$, we can cross multiply and simplify as follows:

$$\begin{aligned} 2x^2 - 40 &= x^2 + 20x + 4 \\ x^2 - 20x - 44 &= 0 \\ (x - 22)(x + 2) &= 0. \end{aligned}$$

   Therefore, $x = 22$ or $x = -2$.

   Next, we can solve $\frac{x^2-24}{x^2+24x+4} = \frac{1}{2}$ by cross multiplying:

$$\begin{aligned} 2x^2 - 48 &= x^2 + 24x + 4 \\ x^2 - 24x - 52 &= 0 \\ (x - 26)(x + 2) &= 0. \end{aligned}$$

   Here, $x = 26$ or $x = -2$. The only solution to both equations is therefore $x = \boxed{-2}$.

3. Suppose $a_1, a_2, \ldots$ is an arithmetic sequence, and suppose $g_1, g_2, \ldots$ is a geometric sequence with common ratio 2. Suppose $a_1 + g_1 = 1$ and $a_2 + g_2 = 1$. If $a_{24} = g_7$, find a_{2024}.

   Answer: −22

   Solution: Let $g_1 = S$ and $g_2 = 2S$. Then we have $a_1 = 1 - S$ and $a_2 = 1 - 2S$. Suppose the common difference of the arithmetic sequence is d. This means $d = 1 - 2S - (1 - S) = -S$. Now we can solve for $a_{24} = g_7$ as follows:

$$\begin{aligned} g_7 &= 2^6 S = 64S \\ a_{24} &= a_1 + 23d = (1 - S) - 23S = 1 - 24S \\ 64S &= 1 - 24S \\ S &= \frac{1}{88} \end{aligned}$$

   Therefore,

$$a_{2024} = a_1 + 2023d = 1 - S + 2023(-S) = 1 - 2024S = 1 - 2024\left(\frac{1}{88}\right) = 1 - 23 = \boxed{-22}.$$

4. For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n and let $\lceil n \rceil$ be the smallest integer greater than or equal to n. For example, $\lfloor 2.5 \rfloor = 2$ and $\lfloor 2 \rfloor = 2$, while $\lceil 2.5 \rceil = 3$ and $\lceil 2 \rceil = 2$. Find the greatest integer x such that $\left\lfloor \frac{x}{20} + 20 \right\rfloor = \left\lceil \frac{x}{24} + 24 \right\rceil$.

Answer: 699

Solution: We can simplify the equation as follows:

$$\left\lfloor \frac{x}{20} \right\rfloor + 20 = \left\lceil \frac{x}{24} \right\rceil + 24$$

$$\left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil = 4.$$

Note that $a \le \lceil a \rceil < a + 1$ for any real a, and similarly $a - 1 < \lfloor a \rfloor \le a$ as well. We then have that

$$\frac{x}{20} - 1 - \left(\frac{x}{24} + 1\right) < \left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil = 4 \le \frac{x}{20} - \frac{x}{24}$$

yielding the bounds

$$4 \le \frac{x}{20} - \frac{x}{24} < 6$$

Since $\frac{x}{20} - \frac{x}{24} = \frac{x}{120}$, we have $480 \le x < 720$. We can now go backwards from 720 to find the solution. We can decrease $\left\lfloor \frac{x}{20} \right\rfloor - \left\lceil \frac{x}{24} \right\rceil$ by 1 by decreasing x by 1, which will reduce the floor expression by 1. To decrease by another increment of 1, we will need to decrease x by another 20. Therefore, $x = 720 - 20 - 1 = 699$ gives us

$$\left\lfloor \frac{699}{20} \right\rfloor - \left\lceil \frac{699}{24} \right\rceil = 34 - 30 = 4$$

and our answer is $\boxed{699}$.

5. An ordered pair (a, b) of real numbers is *Z-nice* if $x^3 + ax + b$ has 3 distinct roots p, q, r such that $|p - 2024| = |q - 2024| = |r - 2024| = Z$. Find the greatest possible real value of Z such that there is exactly one Z-nice ordered pair.

Answer: 6072

Solution: Let $x^3 + ax + b$ have roots p, q, r. Note that if p, q, r are real, then it is impossible for $|p - 2024| = |q - 2024| = |r - 2024| = Z$ without having nondistinct p, q, r, so we must have at least one complex root. Then by the Conjugate Root Theorem, two of the roots are complex conjugates; without loss of generality let $q = c + di$ and $r = c - di$. By Vieta's formulas, $p + q + r = 0$, so $p + (c + di) + (c - di) = p + 2c = 0$ and so $p = -2c$.

Furthermore, each root lies on the circle in the complex plane with center $2024 + 0i$ and radius Z. The only possible values for the real root p are $2024 + Z$ and $2024 - Z$. Then $c = -\frac{1}{2}p$, so the only possible combinations of values for p and c are $(2024 + Z, -1012 - Z/2)$ and $(2024 - Z, -1012 + Z/2)$. For these pairs (p, c) to lead to a unique polynomial and thus a Z-nice ordered pair, c must be a real component of at least two points on the circle. That is, c must be in the interval $(2024 - Z, 2024 + Z)$.

If $c = -1012 - Z/2$, then $c \in (2024 - Z, 2024 + Z)$ if $Z > 3 \cdot 2024 = 6072$. If $c = -1012 + Z/2$, then $c \in (2024 - Z, 2024 + Z)$ if $Z > 2024$. Therefore, there is exactly one Z-nice ordered pair when $2024 < Z \le 6072$, so the greatest possible value of Z is $\boxed{6072}$.

6. There exist nonzero real numbers B, M, and T that satisfy the equations:

$$2B+M+T-2B^2-2BM-2MT-2BT=0,$$
$$B+2M+T-3M^2-3BM-3MT-3BT=0,$$
$$B+M+2T-4T^2-4BM-4MT-4BT=0.$$

Compute $2B+3M+4T$.

Answer: 3

Solution: First, rewrite as:

$$2B+M+T=2B^2+2BM+2MT+2BT,$$
$$B+2M+T=3M^2+3BM+3MT+3BT,$$
$$B+M+2T=4T^2+4BM+4MT+4BT.$$

Upon factoring:

$$2B+M+T=2(B+M)(B+T),$$
$$B+2M+T=3(B+M)(M+T),$$
$$B+M+2T=4(B+T)(M+T).$$

Let $x=B+M$, $y=B+T$, and $z=M+T$. Then we have:

$$x+y=2xy,$$
$$x+z=3xz,$$
$$y+z=4yz.$$

Dividing the first equation by xy, the second equation by xz, and the third equation by yz, we then have:

$$\frac{1}{y}+\frac{1}{x}=2,$$
$$\frac{1}{z}+\frac{1}{x}=3,$$
$$\frac{1}{z}+\frac{1}{y}=4.$$

Solving this system gives $(x,y,z)=(2,\frac{2}{3},\frac{2}{5})$. Now to solve for B, M, and T:

$$B+M=2,$$
$$B+T=\frac{2}{3},$$
$$M+T=\frac{2}{5}.$$

Solving this system gives $(B,M,T)=(\frac{17}{15},\frac{13}{15},-\frac{7}{15})$. Therefore, our answer is $2B+3M+4T=2(\frac{17}{15})+3(\frac{13}{15})+4(-\frac{7}{15})=\boxed{3}$.

Alternatively, once finding x,y,z you may obtain that $2B+3M+4T=\frac{3}{2}(x+y+z)-x+z=\frac{3}{2}(2+\frac{2}{3}+\frac{2}{5})-2+\frac{2}{5}=\boxed{3}$.

7. Compute the number of positive integer triples (B, M, T) satisfying $B, M, T < 24$ and

$$BM + MT + BT = (B + M + T)\sqrt[3]{BMT}.$$

Answer: 89

Solution: Let B, M, T be the roots of a polynomial

$$x^3 - px^2 + qx - r = (x - B)(x - M)(x - T).$$

We have $p = B + M + T$, $q = BM + MT + BT$, and $r = BMT$ by Vieta's formulae. Thus, we see that the equation can be rewritten as $q = p\sqrt[3]{r}$, which is equivalent to $q^3 = p^3 r$, or $r = \frac{q^3}{p^3}$. Now, notice that $x = \frac{q}{p}$ is a root of $x^3 - px^2 + qx - r$ because

$$\left(\frac{q}{p}\right)^3 - p\left(\frac{q}{p}\right)^2 + q\left(\frac{q}{p}\right) - r = 0.$$

Therefore,

$$\left(\frac{q}{p} - B\right)\left(\frac{q}{p} - M\right)\left(\frac{q}{p} - T\right) = 0$$
$$(q - pB)(q - pM)(q - pT) = 0$$
$$(MT - B^2)(BT - M^2)(BM - T^2) = 0,$$

after plugging in $p = B + M + T$, $q = BM + MT + BT$, and $r = BMT$.

Thus, we have three cases: $B^2 = MT$, $M^2 = BT$, or $T^2 = BM$. This means B, M, T forms a geometric sequence in some order.

For $B, M, T < 24$, we have the following triples which can be rearranged in any way: $(1, 2, 4)$, $(1, 3, 9)$, $(1, 4, 16)$, $(2, 4, 8)$, $(2, 6, 18)$, $(3, 6, 12)$, $(4, 6, 9)$, $(4, 8, 16)$, $(5, 10, 20)$, $(8, 12, 18)$, and $(9, 12, 16)$.

Since there are 11 triples that can be arranged in any way, there are actually $11 \cdot 3! = 66$ total possibilities when B, M, T are distinct. However, we should not forget the trivial case where $B = M = T$. Thus, we have another 23 cases to add.

Therefore, the total number of triples (B, M, T) is $66 + 23 = \boxed{89}$.

8. Let α be a positive real number. Over all choices of positive real numbers w, x, y, z satisfying

$$wx + yz = \alpha,$$
$$wy + xz = \alpha,$$
$$wz + xy = \alpha,$$

the minimum value of $w + 2x + 3y + 4z$ is $\frac{\alpha}{2}$. Compute α.

Answer: 128

Solution: Multiplying both sides of the first equation by wx and rearranging gives $wxyz = \alpha wx - (wx)^2$. This similarly applies to wy, wz, xy, yz, and yz in the equations each expression appears in, so

$$wxyz = \alpha wx - (wx)^2 = \alpha wy - (wy)^2 = \alpha wz - (wz)^2$$
$$= \alpha xy - (xy)^2 = \alpha xz - (xz)^2 = \alpha yz - (yz)^2.$$

Then wx, wy, wz, xy, xz, and yz are all roots of the quadratic $f(a) = -a^2 + \alpha a - wxyz$, which has at most two distinct real roots. Therefore, among wx, wy, and wz, at least two of them are equal. Since w is nonzero, there is an equal pair among x, y, and z.

Applying this logic with the other triplets amongst w, x, y, and z and combining it all together, we get that either there are three equal numbers or two pairs of equal numbers among w, x, y, and z. If $w = x$ and $y = z$, then the equations gives $w^2 + y^2 = \alpha$ and $2wy = \alpha$, but then $0 = w^2 + z^2 - 2wz = (w - z)^2$, so $w = z$ and so all numbers are equal. Therefore, there must be three equal numbers w, x, y, and z.

Suppose $x = y = z$. Then all equations become $wz + z^2 = \alpha$, and solving for w gives $w = \dfrac{\alpha - z^2}{z}$. So, the general solutions are of the form $\left(z, z, z, \dfrac{\alpha - z^2}{z}\right)$ and its permutations. Then, depending on the permutation, $w + 2x + 3y + 4z$ is one of the following:

$$6z + \frac{4(\alpha - z^2)}{z} = 2z + \frac{4\alpha}{z},$$
$$7z + \frac{3(\alpha - z^2)}{z} = 4z + \frac{3\alpha}{z},$$
$$8z + \frac{2(\alpha - z^2)}{z} = 6z + \frac{2\alpha}{z},$$
$$9z + \frac{\alpha - z^2}{z} = 8z + \frac{\alpha}{z}.$$

Applying AM-GM to each of these expressions, we get that their minimums are $2\sqrt{2 \cdot 4\alpha}$, $2\sqrt{4 \cdot 3\alpha}$, $2\sqrt{6 \cdot 2\alpha}$, and $2\sqrt{8 \cdot \alpha}$. Out of these four values, we see the minimum is achieved at $2\sqrt{2 \cdot 4\alpha} = 2\sqrt{8 \cdot \alpha}$. We have:

$$2\sqrt{8\alpha} = \frac{\alpha}{2}$$
$$4\sqrt{8\alpha} = \alpha$$
$$\alpha^2 - 128\alpha = 0$$
$$\alpha = 0, 128.$$

We know w, x, y, z are positive, so $\alpha = \boxed{128}$.

9. Define two sequences of real numbers, $\{a_n\}$ and $\{b_n\}$, such that $a_0 = b_0 = \sqrt{3}$ and for $n \geq 0$:

$$a_{n+1} = a_n + \sqrt{1 + a_n^2}$$
$$b_{n+1} = \frac{b_n}{1 + \sqrt{1 + b_n^2}}.$$

Find the smallest real number M such that $M > |a_i b_i - a_j b_j|$ for any integers $i, j > 0$.

Answer: $\mathbf{\frac{2\sqrt{3}}{3} - 1}$

Solution: Let $a_n = \cot(\theta_n)$ for $n \geq 1$ and $\theta_n \in \left[0, \frac{\pi}{2}\right]$. Clearly, we have $\theta_1 = \frac{\pi}{6}$, since $\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$. Therefore, we get

$$a_{n+1} = \cot(\theta_n) + \csc(\theta_n) = \cot\left(\frac{\theta_n}{2}\right)$$

using the half-angle identity for cotangent as shown below:

$$\cot\left(\frac{\theta_n}{2}\right) = \frac{1+\cos(\theta_n)}{\sin(\theta_n)}.$$

This identity may be derived using the half-angle identity for tangent as follows:

$$\tan\left(\frac{\theta_n}{2}\right) = \frac{\sin\left(\frac{\theta_n}{2}\right)}{\cos\left(\frac{\theta_n}{2}\right)} = \frac{2\sin\left(\frac{\theta_n}{2}\right)\sin\left(\frac{\theta_n}{2}\right)}{2\sin\left(\frac{\theta_n}{2}\right)\cos\left(\frac{\theta_n}{2}\right)} = \frac{2\sin^2\left(\frac{\theta_n}{2}\right)}{\sin\left(\theta_n\right)} = \frac{1-\cos\left(\theta_n\right)}{\sin\left(\theta_n\right)}.$$

Similarly, we can let $b_n = \tan(2\theta_n)$, since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. We can alternatively write this as

$$b_n = \tan(2\theta_n) = \frac{\sin(2\theta_n)}{\cos(2\theta_n)} = \frac{2\cos(\theta_n)\sin(\theta_n)}{\cos^2(\theta_n)-\sin^2(\theta_n)} = \frac{\frac{2\cos(\theta_n)\sin(\theta_n)}{\cos^2(\theta_n)}}{\frac{\cos^2(\theta_n)-\sin^2(\theta_n)}{\cos^2(\theta_n)}} = \frac{2\tan(\theta_n)}{1-\tan^2(\theta_n)},$$

a derivation for the double-angle formula for tangent.

It follows that

$$b_{n+1} = \frac{\tan(\theta_n)}{1+\sec(2\theta_n)} = \frac{\frac{\sin(2\theta_n)}{\cos(2\theta_n)}}{1+\frac{1}{\cos(2\theta_n)}} = \frac{\sin(2\theta_n)}{\cos(2\theta_n)+1} = \frac{2\sin(\theta_n)\cos(\theta_n)}{2\cos^2(\theta_n)} = \frac{\sin(\theta_n)}{\cos(\theta_n)} = \tan(\theta_n).$$

Notice how, for both sequences a_n and b_n, θ_n is a geometric sequence with common ratio $\frac{1}{2}$. Since $\theta_1 = \frac{\pi}{6}$, we have $\theta_n = \frac{\pi}{6(2)^{n-1}}$. From this, we get $\theta_n < \frac{\pi}{6}$.

The problem is asking us to find the difference between the least upper bound and greatest lower bound of $a_n b_n$, where

$$a_n b_n = \cot(\theta_n)\tan(2\theta_n) = \frac{2}{1-\tan^2(\theta_n)}.$$

We know that $\tan^2(\theta_n) \in (0,1)$, so we can say the greatest lower bound is $a_n b_n > 2$.

To upper bound this, notice that as n gets larger, θ_n gets smaller, with its maximum achieved at $\theta_1 = \frac{\pi}{6}$. Thus, $\tan^2(\theta_n) < \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}$, which implies $a_n b_n < 3$.

However, notice that this upper bound is the value of $a_0 b_0$, which is not included in our sequence. As such, since this is a decreasing function for $n \to \infty$, the smallest upper bound we can achieve is $a_n b_n \le a_1 b_1 = \left(\sqrt{3}+2\right)\left(\frac{\sqrt{3}}{3}\right) = 1 + \frac{2\sqrt{3}}{3}$.

Thus, our answer is $M = 1 + \frac{2\sqrt{3}}{3} - 2 = \boxed{\frac{2\sqrt{3}}{3} - 1}$.

10. Let $\omega = e^{\frac{2\pi i}{5}}$, where $i = \sqrt{-1}$. If

$$S = \prod_{0 \le a,b,c < 5} \left(\omega^a - \sec\left(\frac{2\pi(b-a)}{5}\right)\cdot \omega^b + \omega^c\right),$$

compute the remainder when S is divided by 101.

Answer: 91

Solution: We transform the given product by multiplying each factor by ω^{-a}. There are 5^2 terms starting with ω^a, so the total multiplication is by $\omega^{-25Ka} = 1$ for some integer K. Therefore,

$$\prod_{0\le a,b,c<5}\left(\omega^a + \sec\frac{2\pi(b-a)}{5}\cdot\omega^b + \omega^c\right) = \prod_{0\le a,b,c<5}\left(1 - \sec\frac{2\pi(b-a)}{5}\cdot\omega^{b-a} + \omega^{c-a}\right)$$

For any fixed a, we can choose b and c so that $b-a$ and $c-a$ have a specific residue modulo 5. Since $\omega^k = \omega^{k+5x}$ for any integers k, x, we can conclude that

$$\prod_{0\le a,b,c<5}\left(1 - \sec\frac{2\pi(b-a)}{5}\cdot\omega^{b-a} + \omega^{c-a}\right) = \left(\prod_{0\le a,b<5}(1 - \sec\frac{2\pi a}{5}\cdot\omega^a + \omega^b)\right)^5$$

We then want to simplify

$$S^{\frac{1}{5}} = \prod_{0\le a,b<5}\left(1 - \sec\frac{2\pi a}{5}\cdot\omega^a + \omega^b\right),$$

which we can do by using the fact that $x^5+1 = (x+1)(x+\omega)(x+\omega^2)(x+\omega^3)(x+\omega^4)$. We can use this factoring over ω^b and substitute $x = 1-\omega^a\sec\frac{2\pi a}{5}$ to obtain

$$\prod_{0\le a,b<5}\left(1 - \sec\frac{2\pi a}{5}\cdot\omega^a + \omega^b\right) = \prod_{0\le a<5}\left(\left(1-\omega^a\sec\frac{2\pi a}{5}\right)^5 + 1\right)$$

Substituting $\omega^a = e^{\frac{2\pi a i}{5}} = \cos\frac{2\pi a}{5} + i\sin\frac{2\pi a}{5}$ yields

$$1-\omega^a\sec\frac{2\pi a}{5} = 1 - \frac{\cos\frac{2\pi a}{5} + i\sin 2\pi a5}{cos\frac{2\pi a}{5}} = -i\tan\frac{2\pi a}{5}$$

which allows us to transform the product as

$$\prod_{0\le a<5}\left(\left(1-\omega^a\sec\frac{2\pi a}{5}\right)^5+1\right) = \prod_{0\le a<5}\left(\left(-i\tan\frac{2\pi a}{5}\right)^5+1\right)$$

Factoring out a $\sec^5\frac{2\pi a}{5}$ from each term gives

$$\prod_{0\le a<5}\left(\left(-i\tan\frac{2\pi a}{5}\right)^5+1\right) = \left(\prod_{0\le a<5}\cos\frac{2\pi a}{5}\right)^{-5}\cdot\prod_{0\le a<5}\left(\cos^5\frac{2\pi a}{5} - i\sin^5\frac{2\pi a}{5}\right)$$

The left product is equal to 2^{20}; this may be computed by taking

$$\prod_{0\le a<5}\cos\frac{2\pi a}{5} = \prod_{0\le a<5}\frac{\omega^a+\omega^{-a}}{2} = 2^{-5}\prod_{0\le a<5}(1+\omega^{2a}) = 2^{-5}\cdot 2\cdot(1^4-1^3+1^2-1+1) = 2^{1-5}$$

and then raising to the power of -5.

From there, we need to compute

$$2^{20}\prod_{0\le a<5}\left(\cos^5\frac{2\pi a}{5} - i\sin^5\frac{2\pi a}{5}\right)$$

which we do by considering the fact that $\cos\frac{2\pi a}{5} = \frac{\omega^a+\omega^{-a}}{2}$ and $i\sin\frac{2\pi a}{5} = \frac{\omega^a-\omega^{-a}}{2}$. Plugging in and computing gives

$$\begin{aligned}\cos^5\frac{2\pi a}{5} - i\sin^5\frac{2\pi a}{5} &= \left(\frac{\omega^a+\omega^{-a}}{2}\right)^5 - \left(\frac{\omega^a-\omega^{-a}}{2}\right)^5\\ &= 2^{-5}\cdot\left((\omega^{2a}+1)^5 - (\omega^{2a}-1)^5\right)\\ &= 2^{-5}\cdot 2\left(\binom{5}{4}(\omega^{2a})^4 + \binom{5}{2}(\omega^{2a})^2 + \binom{5}{0}\right)\\ &= 2^{-4}\cdot(5\omega^{8a}+10\omega^{4a}+1)\end{aligned}$$

Multiplying each term by ω^{2a} in the product (which multiplies the whole product by $\omega^{2\cdot 10}=1$) gives

$$2^{20}\prod_{0\le a<5}\left(\cos^5\frac{2\pi a}{5} - i\sin^5\frac{2\pi a}{5}\right) = 2^{20-20}\prod_{0\le a<5}(5+10\omega^a+\omega^{2a})$$

We factor the polynomial $x^2+10x+5 = (x-(-5+2\sqrt{5}))(x-(-5-2\sqrt{5})) = (x+(5-2\sqrt{5}))(x+(5+2\sqrt{5}))$. Finally, we can rewrite the product as

$$\begin{aligned}\prod_{0\le a<5}(5+10\omega^a+\omega^{2a}) &= \prod_{0\le a<5}(\omega^a+(5-2\sqrt{5}))(\omega^a+(5+2\sqrt{5}))\\ &= \prod_{0\le a<5}(\omega^a+(5-2\sqrt{5}))\cdot\prod_{0\le a<5}(\omega^a+(5+2\sqrt{5}))\\ &= ((5-2\sqrt{5})^5+1)((5+2\sqrt{5})^5+1)\end{aligned}$$

with the last transformation again coming from the fact that $x^5+1 = (x+1)(x+\omega)(x+\omega^2)(x+\omega^3)(x+\omega^4)$. We then expand this out, precomputing $5^2 = 25, 5^3 = 125 \equiv 24, 5^4 \equiv 24\cdot 5 \equiv 19, 5^5 \equiv 19\cdot 5 \equiv 95 \equiv -6$:

$$\begin{aligned}((5-2\sqrt{5})^5+1)((5+2\sqrt{5})^5+1) &= (25-20)^5+1+(5-\sqrt{20})^5+(5+\sqrt{20})^5\\ &= 5^5+1+2(5^5+10\cdot 5^3\cdot 20+5\cdot 5\cdot 20^2)\\ &= 5^5+1+2(5^5+8\cdot 5^5+25\cdot 400)\\ &\equiv -6+1+2(-6+-6\cdot 8+25\cdot -4) \pmod{101}\\ &\equiv 91 \pmod{101}\end{aligned}$$

Recall that this is the value of $S^{\frac{1}{5}}$, so we must compute $91^5 \pmod{101}$. Since $100 = (-10)^2 \equiv -1 \pmod{101}$, we have $91^5 \equiv (-1)^2\cdot 91 = 91 \pmod{101}$. Therefore, $S \equiv \boxed{91} \pmod{101}$.