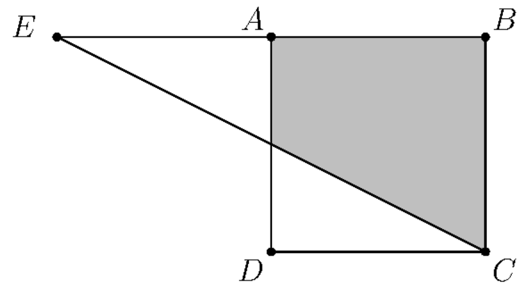
1. Given a square $ABCD$ of side length 6, the point $E$ is drawn on the line $\overline{AB}$ such that the distance $EA$ is less than $EB$ and the triangle $\triangle BCE$ has the same area as $ABCD$. Compute the shaded area.




**Answer: 27**

**Solution:** Let the intersection of $\overline{CE}$ and $\overline{AD}$ be $F$. The area of triangle $\triangle BCE$ is the shaded area plus the area of triangle $\triangle AFE$, which is similar to $\triangle BCE$ with half the side length. Therefore, the area of $\triangle AFE$ is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ the area of $\triangle BCE$, which is equal to the area of square $ABCD$, or $6^2 = 36$. So, the shaded area is $(1 - \frac{1}{4}) \cdot 36 = \boxed{27}$.

2. Jerry has red blocks, yellow blocks, and blue blocks. He builds a tower 5 blocks high, without any 2 blocks of the same color touching each other. Also, if the tower is flipped upside-down, it still looks the same. Compute the number of ways Jerry could have built this tower.

**Answer: 12**

**Solution:** We go step-by-step placing blocks until the whole tower is decided. The bottom block can be any of the 3 colors. Then, the next 2 blocks have 2 options each for what color they can be: the only option ruled out is the color directly below it in our stack. This gives $3 \cdot 2^2 = 12$ for the first 3 blocks in the stack. Then, the remaining 2 blocks are fixed as flipping the tower over tells us they are the same as the bottom 2 blocks. So, there are $\boxed{12}$ possible towers.

3. Compute the second smallest positive whole number that has exactly 6 positive whole number divisors (including itself).

**Answer: 18**

**Solution:** It is possible to just check all numbers up to 18 until you see that 12 is the first number with 6 factors and 18 is the second, but there is a faster approach. The number of divisors of a number is equal to the product of the number of choices you have for each prime factor. For example, take $60 = 2^2 \cdot 3 \cdot 5$. For a given divisor, you can take $2^0, 2^1$, or $2^2$ as the power of 2, $3^0, 3^1$ as the power of 3, and $5^0, 5^1$ as the power of 5. So, you have $3 \cdot 2 \cdot 2 = 12$ choices of divisor for 60.

Since we care about numbers with 6 divisors, we want to check for numbers that are written like $p^2q$ or $p^5$ for primes $p, q$, so that we either have $3 \cdot 2$ choices or just 6 choices outright. We use the smallest primes we can, 2 and 3, to get $2^2 \cdot 3 = 12$ as the smallest number with 6 factors. Then we try $3^2 \cdot 2 = 18$, which is in fact the next smallest. We also check $2^2 \cdot 5 = 20 > 18$ and $2^5 = 32 > 18$, after which everything is definitely bigger than 18. This gives us our final answer of $\boxed{18}$.

4. A grasshopper is traveling on the coordinate plane, starting at the origin $(0,0)$. Each hop, the grasshopper chooses to move 1 unit up, down, left, or right with equal probability. The grasshopper hops 4 times and stops at point $P$. Compute the probability that it is possible to return to the origin from $P$ in at most 3 hops.

**Answer:** $\mathbf{\frac{49}{64}}$

**Solution:** We will compute the complement (that it will take exactly 4 hops to return to the origin) and then subtract from 1. Notice that if we ever hop in more than 2 distinct directions, one will be the opposite of the other direction and will "cancel" out, allowing us to return from $P$ in less than 4 hops. Therefore, we must select at most 2 directions that we may hop in. However, these directions can't be up and down or left and right as these are opposite directions. So, there are 4 ways of choosing the pairs of 2 directions the grasshopper can hop in.

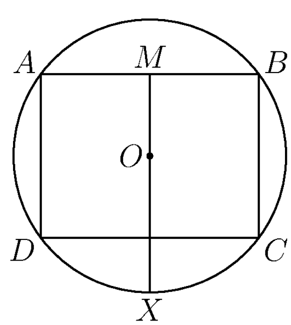
Then, each hop is selected from one of those 2 directions for each of the pairs, which gives $2^4$ possible hops for each of the 4 pairs and a total count of $4(2^4)$. However, we count each of the cases where we hop only in one direction exactly twice, and there are 4 of those cases. So we should subtract 4 from our total to get $4(2^4 - 1)$ possible ways. We divide by the total number of ways of hopping which is just $4^4$ giving the result $\frac{4(15)}{4^4} = \frac{15}{64}$. Our answer is the complement: $1 - \frac{15}{64} = \boxed{\frac{49}{64}}$.

5. Two parabolas, $y = ax^2 + bx + c$ and $y = -ax^2 - bx - c$, intersect at $x = 2$ and $x = -2$. If the $y$-intercepts of the two parabolas are exactly 2 units apart from each other, compute $|a + b + c|$.

**Answer:** $\mathbf{\frac{3}{4}}$

**Solution:** First, notice that the two parabolas are reflections of each other across the $x$-axis since, for every $x$, the sum of the $y$ values is 0. So, the two parabolas intersect whenever their $y$ values are 0, meaning that they both pass through $(-2, 0)$, and $(2, 0)$. Then, from symmetry across the line $y = 0$, we get that the $y$-intercepts of the two parabolas are $(0, \pm 1)$. In order to solve for $a, b$, and $c$, we solve for one of the parabolas and note that the other parabola simply has all of its coefficients negated, meaning that $|a + b + c|$ won't be affected. From $(0, -1)$, we get that $a(0)^2 + b(0) + c = -1$, or $c = -1$. Then, from $(-2, 0)$ and $(2, 0)$ and plugging in $-1$ for $c$, we get that $0 = a(-2)^2 + b(-2) - 1 = 4a^2 - 2b - 1$ and $0 = a(2)^2 + b(2) - 1 = 4a^2 + 2b - 1$, so $a = \frac{1}{4}$ and $b = 0$. Therefore, our answer is $|a + b + c| = \left|\frac{3}{4}\right| = \boxed{\frac{3}{4}}$.

6. Let rectangle $ABCD$ have side lengths $AB = 8, BC = 6$. Let $ABCD$ be inscribed in a circle with center $O$, as shown in the diagram. Let $M$ be the midpoint of side $\overline{AB}$, and let $X$ be the intersection of ray $\overrightarrow{MO}$ with the circle. Compute the length $AX$.



**Answer: $4\sqrt{5}$**

**Solution:** Since $AB$ is a chord of circle $O$ and $M$ is the midpoint, we have triangle $\angle AMX = 90^\circ$. Solving for lengths, $AM = \frac{AB}{2} = \frac{8}{2} = 4$, $MO = \frac{BC}{2}$, and $OX = \sqrt{3^2+4^2} = 5$ by the Pythagorean Theorem since $\overline{OX}$ is the radius of the circle. So, $MX = MO + OX = 3 + 5 = 8$. Finally, by the Pythagorean Theorem, $AX^2 = AM^2 + MX^2 = 4^2 + 8^2 = 80$, so $AX = \sqrt{80} = \boxed{4\sqrt{5}}$.

7. For an integer $n > 0$, let $p(n)$ be the product of the digits of $n$. Compute the sum of all integers $n$ such that $n - p(n) = 52$.

**Answer: 157**

**Solution:** Let $n$ be a natural number with $d$ digits. We represent $n$ as a number in base 10 in the following way:

$$n = a_{d-1}10^{d-1} + a_{d-2}10^{d-2} + \cdots + a_1 10^1 + a_0,$$

where $a_{d-1}$ is an integer 1 to 9 and $a_j$ is an integer 0 to 9 for all $0 \le j < d-1$.

Over all $d$ digit possibilities for $n$ with leading digit $a_{d-1}$, we know that $p(n)$ would be maximized if all $d$ digits of $n$ were 9, so we can set an upper bound for $p(n)$ as $9^{d-1}a_{d-1}$. Also, we can get a lower bound for any $n$ by simply taking the first term of the base 10 expansion: $a_{d-1}10^{d-1}$. Therefore, we can only have solutions when $10^{d-1}a_{d-1} - 9^{d-1}a_{d-1} \le 52$, as otherwise we would have $n - p(n) > 52$. We can rewrite the inequality as $a_{d-1}(10^{d-1} - 9^{d-1}) < 52$, which holds for $d \le 2$, and for $d = 3$ if $a_2 < 3$. It's fairly simple to see that this inequality doesn't hold for $d > 3$. Therefore we only have to consider integers of the following forms: $\overline{a}, \overline{ab}, \overline{1ab}, \overline{2ab}$. We consider these cases individually:

- The single-digit case has $p(n) = n$, so this doesn't work.
- The two-digit case looks like this: $10a + b - ab = 52$, which we can factor like this: $ab - 10a - b + 10 = -42 \implies (a-1)(10-b) = 42$. Both of these numbers are between 0 and 9 inclusive, so we have the single factor pair $6 \cdot 7$ as a possibility. here are two ways to assign 6 and 7 to the two factors. First, for $(a-1) = 7, (10-b) = 6$ we get the solution 84, then $(a-1) = 6, (10-b) = 7$ gives 73 as a second solution.
- The first three-digit restricted case looks like this: $100 + 10a + b = ab + 52$. We may factor: $100 + (10-b)a + b = 52$, which is impossible as $100 + (10-b)a + b \ge 100$ since both $a$ and $b$ are between 0 and 9. Therefore, there are no solutions of the form $\overline{1ab}$.
- Similarly, we have $200 + 10a + b = 2ab + 52$, factoring as $200 + (10-2b)a + b = 52$. We have $(10-2b)a > (-100)$ by taking absolute values, so we get $200 + (10-2b)a + b > 100 + b > 52$, so there are no solutions of the form $\overline{2ab}$ either.

Therefore, the only solutions come in the two digit case, giving us the answer $84 + 73 = \boxed{157}$.

8. Circle $\omega_1$ is centered at $O_1$ with radius 3, and circle $\omega_2$ is centered at $O_2$ with radius 2. Line $\ell$ is tangent to $\omega_1$ and $\omega_2$ at $X$, $Z$, respectively, and intersects segment $\overline{O_1O_2}$ at $Y$. The circle through $O_1$, $X$, $Y$ has center $O_3$, and the circle through $O_2$, $Y$, $Z$ has center $O_4$. Given that $O_1O_2 = 13$, find $O_3O_4$.

**Answer: $\frac{13}{2}$**

**Solution:** Because line $\ell$ is tangent to the circles, $\overline{O_1X}$ and $\overline{O_2Z}$ are both perpendicular to $\ell$. So, $\triangle O_1XY$ and $\triangle O_2YZ$ both have right angles at the tangency points. Because the circumcircle of a right triangle has its center at the midpoint of the hypotenuse of the right triangle, $O_3$ and $O_4$ are the midpoints of $\overline{O_1Y}$ and $\overline{O_2Y}$, respectively. Thus, $O_3O_4 = \frac{O_1O_2}{2} = \boxed{\frac{13}{2}}$.

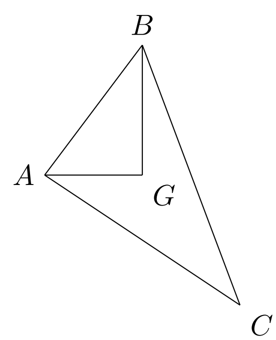
9. For positive integers $a$ and $b$, consider the curve $x^a + y^b = 1$ over real numbers $x, y$ and let $S(a,b)$ be the sum of the number of $x$-intercepts and $y$-intercepts of this curve. Compute $\sum_{a=1}^{10}\sum_{b=1}^{5} S(a,b)$.

**Answer: 145**

**Solution:** We first notice that $S(a,b)$ is just dependent on the values of $a$ and $b$ modulo 2. In particular, the number of $x$-intercepts is the number of solutions to $x^a = 1$, which is 1 if $a$ is odd and 2 if $a$ is even. Let's call this function $X(a)$. The logic holds for the $y$-intercepts depending on $b$: the number of $y$-intercepts in terms of $b$ we can write as $Y(b) = X(b)$. Therefore, we can rewrite the sum as

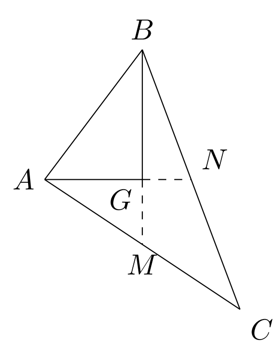
$$\sum_{a=1}^{10}\sum_{b=1}^{5}(X(a)+Y(b)) = \sum_{a=1}^{10} 3(X(a)+1)+2(X(a)+2) = \sum_{a=1}^{10} 5X(a)+7 = 70+5(5\cdot 1+5\cdot 2) = \boxed{145}.$$

10. Let $\triangle ABC$ be a triangle with $G$ as its centroid, which is the intersection of the three medians of the triangle, as shown in the diagram. If $\overline{GA} \perp \overline{GB}$ and $AB = 7$, compute $AC^2 + BC^2$.




**Answer: 245**

**Solution:**




Extend $\overrightarrow{BG}$ to $M$ and $\overrightarrow{AG}$ to $N$. Since $G$ is the centroid of $\triangle ABC$, we know that $M$ and $N$ are the midpoints of $\overline{AC}$ and $\overline{BC}$ and that $GM = \frac{1}{2}BG$ and $GN = \frac{1}{2}AG$. Then, we can set up the following equations:

$$\left(\frac{AC}{2}\right)^2 = AM^2 = AG^2 + GM^2,$$
$$\left(\frac{BC}{2}\right)^2 = BN^2 = BG^2 + GN^2.$$

By the Pythagorean Theorem, we have that $BG^2 + AG^2 = AB^2 = 7^2 = 49$ and $GM^2 + GN^2 = MN^2 = \left(\frac{AB}{2}\right)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$. Then, by adding the equtions, we can get $AC^2 + BC^2 = 4\left(AG^2 + BG^2 + GM^2 + GN^2\right) = 4\left(49 + \frac{49}{4}\right) = \boxed{245}$.

11. Compute the sum of all positive integers $n$ for which there exists a real number $x$ satisfying

$$\left(x + \frac{n}{x}\right)^n = 2^{20}.$$

**Answer: 36**

**Solution:** Observe that, for a positive number $y$, $x + \frac{n}{x} = y \implies x^2 - yx + n = 0$ has a real solution, $x$, if and only if $y^2 - 4n \geq 0$ by the quadratic formula. In particular, $x + \frac{n}{x} = 2^{20/n}$ has a real solution if and only if $2^{40/n-2} \geq n$. Note that whenever $n \leq 8$, we have $2^{40/n-2} \geq 2^3 \geq n$, and whenever $n > 8$, we have $2^{40/n-2} < 2^3 < n$. This implies that the answer is $1+2+\cdots+8 = \boxed{36}$.

12. Call an $n$-digit integer with distinct digits *mountainous* if, for some integer $1 \leq k \leq n$, the first $k$ digits are in strictly ascending order and the following $n-k$ digits are in strictly descending order. How many 5-digit mountainous integers with distinct digits are there?

**Answer: 3024**

**Solution:** This problem can be solved via casework on $k$, but we present a faster solution. Suppose we have 5 distinct digits $0 \leq d_1 < d_2 < \cdots < d_5 \leq 9$. We have $d_5$ as the largest digit, which must be the "peak" of the mountain. The rest of the digits must either go before or after $d_5$, and, once we've fixed the side that each other digit is on, there is only one way of ordering them so that they are increasing or decreasing (depending on the side). Then we have two cases: $d_1 > 0$, or $d_1 = 0$.

If $d_1 > 0$, then there are $\binom{9}{5}$ ways of picking the 5 digits, combined with $2^4$ ways of choosing which side of the peak each of the 4 digits $d_1, \cdots, d_4$ can go. If $d_1 = 0$, then there are $\binom{9}{4}$ ways of picking the remaining 4 digits, and there are $2^3$ ways of choosing which side of the peak each of the 3 digits $d_2, d_3, d_4$ can go (note that $d_1$ is forced to the right since the leading digit cannot be 0). Thus, the final answer is $2^4 \cdot \binom{9}{5} + 2^3 \cdot \binom{9}{4} = \boxed{3024}$.

13. Consider the set of triangles with side lengths $1 \leq x \leq y \leq z$ such that $x, y$, and $z$ are the solutions to the equation $t^3 - at^2 + bt = 12$ for some real numbers $a$ and $b$. Compute the smallest real number $N$ such that $N > ab$ for any choice of $x, y$, and $z$.

**Answer: 152**

**Solution:** First, we apply Vieta's formulas to get that $ab = (x+y+z)(xy+yz+zx)$. We multiply by $1 = \frac{12}{xyz}$ and expand out to get that

$$ab = 12\left(3 + \frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{z}{y} + \frac{y}{z}\right).$$

WLOG, let $x \le y \le z$. We observe that, by AM-GM, this is maximized when $x, y$, and $z$ are as far apart as possible, so we minimize $x$ by taking $x = 1$. Now, we also want to make $y$ and $z$ as far from each other as possible. Because of the triangle inequality, $x + y > z$, so we want to approach $z = x + y$ to get an upper bound for the possible value of $ab$. So we get the equation $1(y)(y + 1) = xyz = 12$, meaning $y = 3$ and $z = 4$. From here, we can calculate $ab$ explicitly as

$$ab = 12\left(3 + \frac{1}{3} + \frac{3}{1} + \frac{1}{4} + \frac{4}{1} + \frac{3}{4} + \frac{4}{3}\right) = 12\left(\frac{38}{3}\right) = \boxed{152}.$$

14. Right triangle $\triangle ABC$ with $\angle A = 30^\circ$ and $\angle B = 90^\circ$ is inscribed in a circle $\omega_1$ with radius 4. Circle $\omega_2$ is drawn to be the largest circle outside of $\triangle ABC$ that is tangent to both $\overline{BC}$ and $\omega_1$, and circles $\omega_3$ and $\omega_4$ are drawn this same way for sides $\overline{AC}$ and $\overline{AB}$, respectively. Suppose that the intersection points of these smaller circles with the bigger circle are noted as points $D$, $E$, and $F$. Compute the area of triangle $\triangle DEF$.

**Answer: $\mathbf{12 + 4\sqrt{3}}$**

**Solution:** If we say that $O$ is the center of the circumcircle $\omega_1$, noting that $O$ is within $\triangle DEF$, then $[DEF] = [ODE] + [OEF] + [OFD]$. Furthermore, hypotenuse $\overline{AC}$ is a diameter of the circle that $\triangle ABC$ is inscribed in, so $O$ is the midpoint of $\overline{AC}$. Now, in order to maximize the area of the circles externally tangent to the triangle and internally tangent to the circumcircle, $D, E$, and $F$ should be the midpoints of arcs $\widehat{BC}, \widehat{AC}$, and $\widehat{AB}$ (as these are the furthest points on the circumcircle from the triangle). Therefore, $\overline{OD}, \overline{OE}$, and $\overline{OF}$ are angle bisectors of $\angle BOC, \angle COA$, and $\angle AOB$. Therefore, $\angle DOE = \angle DOC + \angle COE = \frac{\angle COB}{2} + \frac{\angle COA}{2} = 30^\circ + 90^\circ = 120^\circ$ and $\angle FOE = \angle FOA + \angle AOE = \frac{\angle BOA}{2} + \frac{\angle AOC}{2} = 150^\circ$, and $\angle EOD = \angle EOB + \angle BOD = \frac{\angle AOB}{2} + \frac{\angle BOC}{2} = 60^\circ + 30^\circ = 90^\circ$. Therefore, $[OED] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin(120^\circ) = 4\sqrt{3}$, $[OEF] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin(150^\circ) = 4$, and $[ODF] = \frac{1}{2} \cdot 4 \cdot 4 \cdot \sin(90^\circ) = 8$. Thus, our final answer is the sum of these three areas, or $\boxed{12 + 4\sqrt{3}}$.

15. Given a positive integer $k$, let $s(k)$ denote the sum of the digits of $k$. Let $a_1, a_2, a_3, \ldots$ denote the strictly increasing sequence of all positive integers $n$ such that $s(7n + 1) = 7s(n) + 1$. Compute $a_{2023}$.

**Answer: 11111100111**

**Solution:** Let $S = \{a_1, a_2, \ldots\}$. We claim that $n \in S$ if and only if $n$ is a positive integer whose digits are only 1s and 0s. Certainly this is sufficient because $7n + 1$ will have no carries in the multiplication and addition, so $s(7n + 1) = 7s(n) + 1$ follows. To see that this is necessary, write

$$n = \sum_{i=0}^{m} d_i 10^i$$

for digits $d_0, \ldots, d_m$. Then $s(a + b) \le s(a) + s(b)$ implies

$$s(7n + 1) = s\left((7d_0 + 1) + \sum_{i=1}^{m} 7d_i 10^i\right) \le 7d_0 + 1 + \sum_{i=1}^{m} 7d_i = 7s(n) + 1$$

with equality if and only if there were no carries in any of the additions. In particular, we require no carries in $s(7d_i) = 7s(d_i)$, which requires $d_i \in \{0, 1\}$.

Thus, the sequence $a_1, a_2, \ldots$ looks just like counting in binary. So the answer will have the same format of 0s and 1s as 2023 written in binary. We write

$$\begin{aligned}2023 &= 2048 - 25\\ &= 2047 - 16 - 8\\ &= 1,11111,11111_2 - 10000_2 - 1000_2\\ &= 1,11111,00111_2,\end{aligned}$$

so our answer is $\boxed{11111100111}$.

16. Sabine rolls a fair 14-sided die numbered 1 to 14 and gets a value of $x$. She then draws $x$ cards uniformly at random (without replacement) from a deck of 14 cards, each of which labeled a different integer from 1 to 14. She finally sums up the value of her die roll and the value on each card she drew to get a score of $S$. Let $A$ be the set of all obtainable scores. Compute the probability that $S$ is greater than or equal to the median of $A$.

**Answer:** $\mathbf{\frac{15}{28}}$

**Solution:** The key insight is to develop the following bijection: for every event where Sabine scores $1 \le S < 119$ points (i.e. whenever $d \neq 14$), there is a corresponding event where she scores $119 - S$ points. Consider a *turn* $t = (d, p)$ of the game where $d$ is the value of the dice roll and $p$ is a permutation of the integers from 1 to 14. The score of this turn $s(t)$ is given by the sum of $d$ and the sum of the first $d$ values of the permutation $p$ (denoted by $p_d$). In short, $s(t) = d + p_d$. Then, for a given turn $t$ scoring $s(t)$ points where $d < 14$, there is a corresponding turn $t' = (14 - d, p')$ where $p'$ is the reverse permutation of $p$. We have $s(t')$ as $(14-d)+(1+2+\cdots+14)-p_d$, as we sum $14-d$ and the remaining $14-d$ integers not counted from the $(p, d)$ score. This totals to

$$s(t') = 14 + (1 + 2 + \cdots + 14) - (d + p_d) = 14 + \frac{14(15)}{2} - s(t) = 119 - s(t)$$

which was what we wanted.

Still ignoring the case where $d = 14$, this means the median score is $\frac{119}{2} = 59.5$ as every score larger than 59.5 is complemented by a score lower than 59.5. Notably, the median is not an obtainable score. When $d = 14$, we get a guaranteed score of $14 + \frac{14(15)}{2} = 119$, as we are guaranteed to draw every card. Adding 119 to $A$ makes the median the smallest achievable score greater than 59.5, which means we just need to find the probability that our score is greater than 59.5. Observe that the bijection also carries over to probabilities: given that we do not roll a 14, we have an equal likelihood of scoring above 59.5 as we do of scoring below it. If we do roll 14, we are of course guaranteed to score higher than 59.5. Thus, the final probability is $\frac{13}{14} \cdot \frac{1}{2} + \frac{1}{14} = \boxed{\frac{15}{28}}$.

17. Let $N$ be the smallest positive integer divisble by $10^{2023} - 1$ that only has the digits 4 and 8 in decimal form (these digits may be repeated). Compute the sum of the digits of $\frac{N}{10^{2023}-1}$.

**Answer: 20234**

**Solution:** Let $f_k(m)$ be the number created by repeating each digit of $m$ consecutively $k$ times. For example, $f_3(253) = 222555333$. Let $d = 10^{2023} - 1$, which is also equal to $f_{2023}(9)$. Because $10^{2023} \equiv 1 \pmod{d}$, if we split $N$ into contiguous (but not overlapping) strings of 2023 digits and sum them, we should get a number with the same remainder $\pmod d$ as $N$. Call this operation $g$. Now, for any positive integer $N_0$ that only has digits divisible by 4, we must have that $g(N_0)$ is divisible by 4. Therefore, if $N_0$ is also divisible by $10^{2023}-1$, the smallest value that $g(N_0)$ can be is $4 \cdot \left(10^{2023}-1\right) = 3\underbrace{99\cdots9}_{2023\text{ 9's}}6$. We can rewrite this as $36 \cdot (\underbrace{11\cdots1}_{2022\text{ 1's}})$. Observe that $4+8+8+8+8 = 36$, and in fact this sum has the minimal number of terms for 36. This shows that the minimum number of digits that we need is at least $5 \times 2023$. Additionally, note that for any other $N_0 < f_{2023}(48888)$ with only digits 4 and 8, $f(N_0)$ simply cannot be large enough: all of the individual digit sums (without rounding) are at most 36, and there is at least one digit sum that is at most 32. Also, $N = f_{2023}(48888)$ is therefore also divisible by $10^{2023}-1$, meaning that the minimal value of $N$ as $N = f_{2023}(48888)$.

Now, we have to divide $N$ by $10^{2023}-1$. Let $n = 2023$ for convenience. We write that

$$
\begin{aligned}
N &= f_n(48888) \\
N &= \frac{4}{9}\left(10^n-1\right)\left(10^{4n}\right) + \frac{8}{9}\left(10^n-1\right)\left(10^{3n}+10^{2n}+10^n+1\right) \\
N &= \left(10^n-1\right)\left(\frac{4}{9}10^{4n} + \left(10^{3n}+10^{2n}+10^n+1\right) - \frac{1}{9}\left(10^{3n}+10^{2n}+10^n+1\right)\right) \\
\frac{N}{10^n-1} &= \frac{4}{9}10^{4n} + \left(10^{3n}+10^{2n}+10^n+1\right) - \frac{1}{9}\left(10^{3n}+10^{2n}+10^n+1\right).
\end{aligned}
$$

We will use this expression to compute the sum of digits, proceeding carefully so that no carries occur in the addition (which would be troublesome to deal with). First, $\frac{4}{9}10^{4n}$ contributes $4 \cdot 4n$ to the total sum of digits of $\frac{N}{10^n-1}$ (ignoring the digits after the decimal point, which will cancel with the fractional part of $\frac{1}{9}\left(10^{3n}+10^{2n}+10^n+1\right)$). Since all of the digits so far are 4, we can now subtract $\frac{1}{9}\left(10^{3n}+10^{2n}+10^n+1\right)$ to remove at most 4 from each digit safely without needing a carry. This subtracts $3n+2n+n = 6n$ from the total sum of digits, leaving us at $10n$. Finally, the terms $(10^{3n}+10^{2n}+10^n+1)$ add $1 \cdot 4$ to the sum of digits each giving us $10n+4$. When we plug in $n = 2023$, we get $10(2023)+4 = \boxed{20234}$.

*Note: Problem 18 as stated in the competition had an incorrect solution. The problem statement has been corrected here: the original statement is maintained at the end of the solution.*

18. Consider the sequence $b_1, b_2, b_3, \ldots$ of real numbers defined by $b_1 = \frac{3+\sqrt{3}}{6}$, $b_2 = 1$, and for $n \geq 3$,

$$b_n = \frac{2b_{n-1}b_{n-2} - 2b_{n-1} - 2b_{n-2} + 1}{4b_{n-1}b_{n-2} - 2b_{n-1} - 2b_{n-2}}.$$

Compute $b_{2023}$.

**Answer:** $\mathbf{\frac{3-\sqrt{3}}{6}}$

**Solution:** We apply a series of substitutions. First, the denominator can be rewritten via Simon's Favorite Factoring Trick:

$$b_n = \frac{2b_{n-1}b_{n-2} - 2b_{n-1} - 2b_{n-2} + 1}{(2b_{n-1}-1)(2b_{n-2}-1) - 1}.$$

We would like to substitute $a_n = 2b_n - 1$ based on this factorization. Inspired by this form on the righthand side, we multiply by 2 and subtract 1 to create

$$2b_n - 1 = \frac{2(2b_{n-1}b_{n-2} - 2b_{n-1} - 2b_{n-2} + 1) - (4b_{n-1}b_{n-2} - 2b_{n-1} - 2b_{n-2})}{(2b_{n-1}-1)(2b_{n-2}-1) - 1}$$

$$2b_n - 1 = \frac{2 - 2b_{n-1} - 2b_{n-2}}{(2b_{n-1} - 1)(2b_{n-2} - 1) - 1} = \frac{(2b_{n-1} - 1) + (2b_{n-2} - 1)}{1 - (2b_{n-1} - 1)(2b_{n-2} - 1)}$$

Now, we may define $a_n = 2b_n - 1$. Then, $a_1 = \frac{1}{\sqrt{3}}$, $a_2 = 1$, and

$$a_n = \frac{a_{n-1} + a_{n-2}}{1 - a_{n-1}a_{n-2}}$$

for $n \geq 2$. We recognize this as the tangent addition formula: taking $\theta_n := \arctan a_n$ yields $\theta_1 = \frac{2\pi}{12}$, $\theta_2 = \frac{3\pi}{12}$, and

$$\tan\theta_n = \frac{\tan\theta_{n-1} + \tan\theta_{n-2}}{1 - \tan\theta_{n-1}\tan\theta_{n-2}} = \tan(\theta_{n-1} + \theta_{n-2}) \implies \theta_n = \theta_{n-1} + \theta_{n-2}$$

for $n \geq 2$. Equivalently, since the period of $\tan\theta$ is $\pi$, it suffices to consider $c_n := \frac{12}{\pi}\theta_n$ modulo 12, where $c_1 = 2$, $c_2 = 3$, and $c_n = c_{n-1} + c_{n-2}$ for $n \geq 2$.

We compute $c_3 = 5$, $c_4 = 8$, $c_5 = 1$, $c_6 = 9$, $c_7 = 10$, $c_8 = 7$, $c_9 = 5$, $c_{10} = 0$, $c_{11} = 5$, $c_{12} = 5$, $c_{13} = 10$, $c_{14} = 3$. Thus, $c_{k+12} = 5c_k \pmod{12}$ and $c_{k+24} = 25c_k \equiv c_k \pmod{12}$. Since the period of $c$ is 24, $c_{2023} \equiv c_7 \equiv 10 \pmod{12}$, so

$$b_{2023} = \frac{a_{2023} + 1}{2} = \frac{\tan\theta_{2023} + 1}{2} = \frac{\tan\frac{10\pi}{12} + 1}{2} = \boxed{\frac{3 - \sqrt{3}}{6}}.$$

*The original problem statement was as follows:*

Consider the sequence $b_1, b_2, b_3, \ldots$ of real numbers defined by $b_1 = \frac{3+\sqrt{3}}{6}$, $b_2 = 1$, and for $n \geq 3$,

$$b_n = \frac{1 - b_{n-1} - b_{n-2}}{2b_{n-1}b_{n-2} - b_{n-1} - b_{n-2}}.$$

Compute $b_{2023}$.

19. Let $N_{21}$ be the answer to question 21. Suppose a jar has $3N_{21}$ colored balls in it: $N_{21}$ red, $N_{21}$ green, and $N_{21}$ blue balls. Jonathan takes one ball at a time out of the jar uniformly at random without replacement until all the balls left in the jar are the same color. Compute the expected number of balls left in the jar after all balls are the same color.

**Answer:** $\mathbf{\frac{108}{73}}$

**Solution 1:** Let $N = N_{21}$ for convenience. Equivalently, suppose we continue drawing balls until the jar is empty, and we are asked to compute the expected value of $X$, the length of the streak of same-colored balls at the end. By symmetry, it is equivalent to compute

$$\begin{aligned}\mathbb{E}[X \mid \text{last ball is red}] &= \sum_{x=1}^{N} \mathbb{P}(X \geq x \mid \text{last ball is red}) \qquad (X \in \mathbb{Z}^+)\\ &= \sum_{x=1}^{N} \frac{\binom{3N-x}{N,N,N-x}}{\binom{3N-1}{N,N,N-1}}\\ &= \frac{1}{\frac{(3N-1)!}{N!N!(N-1)!}} \sum_{x=1}^{N} \frac{(3N-x)!}{N!N!(N-x)!}\end{aligned}$$

$$= \frac{(2N)!(N-1)!}{(3N-1)!}\sum_{y=2N}^{3N-1}\binom{y}{2N} \qquad (y := 3N - x)$$
$$= \frac{(2N)!(N-1)!}{(3N-1)!}\frac{(3N)!}{(2N+1)!(N-1)!} \qquad \text{(Hockey stick)}$$
$$= \frac{3N}{2N+1}.$$

Since $N_{21} = 36$, our answer is $\boxed{\frac{108}{73}}$.

**Solution 2:** We make use of indicator variables. We want to compute the probability that a given ball of some color $c$ appears in the final streak of same-colored balls at the end of the drawing. This means, out of the $2N_{21}+1$ balls consisting of the single ball of color $c$ and the $2N_{21}$ balls of different colors than $c$, the ball of color $c$ must occur last. This has probability $\frac{1}{2N_{21}+1}$. Therefore, by linearity of expectation, we have that our desired answer is $3N_{21} \cdot \frac{1}{2N_{21}+1} = \frac{3N_{21}}{2N_{21}+1}$. Plugging in $N_{21} = 36$ gives the answer $\boxed{\frac{108}{73}}$.

20. Let $N_{19}$ be the answer to question 19. For every non-negative integer $k$, define

$$f_k(x) = x(x-1) + (x+1)(x-2) + \cdots + (x+k)(x-k-1),$$

and let $r_k$ and $s_k$ be the two roots of $f_k(x)$. Compute the smallest positive integer $m$ such that $|r_m - s_m| > 10N_{19}$.

**Answer: 12**

**Solution:** First, we factor:

$$\begin{aligned} f_k(x) &= \sum_{i=0}^{k}(x+i)(x-i-1) \\ &= \sum_{i=0}^{k}(x^2 - x - i(i+1)) \\ &= (k+1)x^2 - (k+1)x - \left(\frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right) \\ &= (k+1)\left(x^2 - x - \frac{k^2+2k}{3}\right). \end{aligned}$$

By the quadratic formula, $|r_k - s_k| = \sqrt{1 + 4 \cdot \frac{k^2+2k}{3}} = \sqrt{1 + 4 \cdot \frac{(k+1)^2-1}{3}}$. We next try to bound:

$$\begin{aligned} &|r_{m-1} - s_{m-1}| \le 10N_{19} < |r_m - s_m| \\ \implies &4 \cdot \frac{m^2-1}{3} \le 100N_{19}^2 - 1 < 4 \cdot \frac{(m+1)^2-1}{3} \\ \implies &m^2 - 1 \le 75N_{19}^2 - \frac{3}{4} < (m+1)^2 - 1 \\ \implies &m^2 \le 75N_{19}^2 + \frac{1}{4} < (m+1)^2. \end{aligned}$$

Next, we invoke our solutions to problems 19 and 21; let $N_i$ denote the answer to problem $i$. Then $N_{19} = \frac{3N_{21}}{2N_{21}+1} = \frac{3N_{20}^2}{2N_{20}^2+4} \in [1, \frac{3}{2})$.

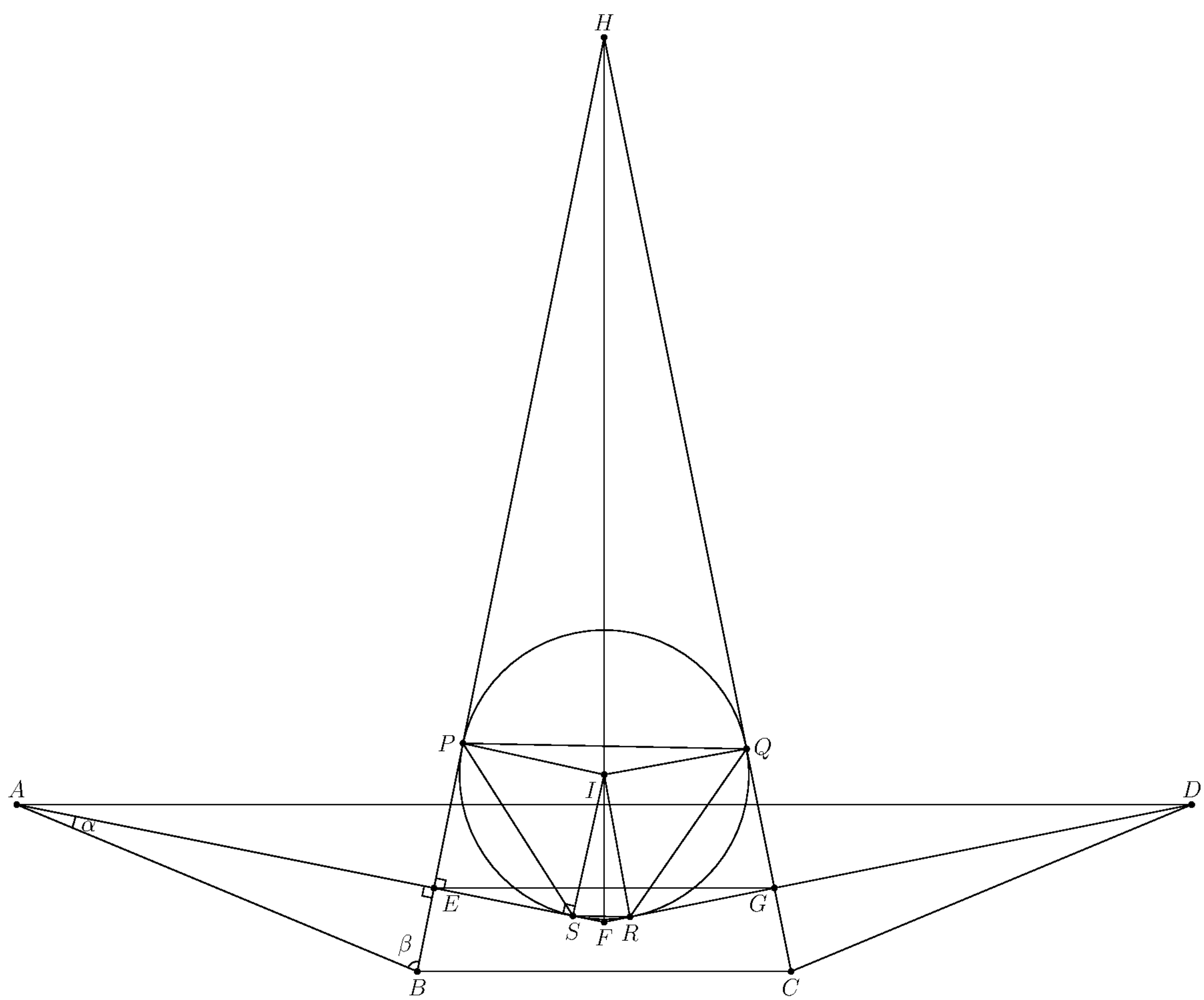
- $N_{19} \geq 1$ implies $(N_{20}+1)^2 > 75 \cdot 1^2 + \frac{1}{4}$, or $N_{20} \geq 8$. Since $N_{19}$ increases with $N_{20}$, it follows that $N_{19} \geq \frac{3 \cdot 8^2}{2 \cdot 8^2+4} = \frac{16}{11} > \sqrt{2}$. Thus, $(N_{20}+1)^2 > 75 \cdot \sqrt{2}^2 + \frac{1}{4} > 144$ implies $N_{20} \geq 12$.
- $N_{19} < \frac{3}{2}$ implies $N_{20}^2 = m^2 < 75 \cdot \left(\frac{3}{2}\right)^2 + \frac{1}{4} = 13^2$, so $N_{20} \leq 12$.

Our final answer is $N_{20} = \boxed{12}$.

21. Let $N_{20}$ be the answer to question 20. In isosceles trapezoid $ABCD$ (where $\overline{BC}$ and $\overline{AD}$ are parallel to each other), the angle bisectors of $A$ and $D$ intersect at $F$, and the angle bisectors of points $B$ and $C$ intersect at $H$. Let $\overline{BH}$ and $\overline{AF}$ intersect at $E$, and let $\overline{CH}$ and $\overline{DF}$ intersect at $G$. If $CG = 3$, $AE = 15$, and $EG = N_{20}$, compute the area of the quadrilateral formed by the four tangency points of the largest circle that can fit inside quadrilateral $EFGH$.

**Answer: 36**

**Solution:**




Let $\alpha = \angle BAE$ and $\beta = \angle ABE$. Then $\alpha + \beta = \frac{1}{2}\angle DAB + \angle ABC = 90^\circ$. By symmetry, $\angle AEB = \angle DGC = 90^\circ$, and $EFGH$ is a kite.

Let the kite's area be $K = \frac{1}{2} \cdot N_{20} \cdot FH$. Since $\angle BHF = \frac{1}{2}(180^\circ - 2\beta) = \alpha$, it follows that $\triangle ABE \sim \triangle HFE$, and the kite's semiperimeter is

$$s = FE + EH = \frac{FH}{AB}(BE + EA) = \frac{3 + 15}{\sqrt{3^2 + 15^2}} FH.$$

Thus, the kite's inradius is $r = K/s = \frac{\sqrt{26}}{12} N_{20}$.

Let the tangency points of the incircle of kite $EFGH$ with lines $\overline{EH}, \overline{HG}, \overline{GF}$, and $\overline{FE}$ be $P$, $Q$, $R$, $S$, respectively, as labeled above. Then, $PISE$ and $QIRG$ are squares with side length $r$. Furthermore, by similar triangles,

$$[IPQ] = [IRS] = \frac{3}{\sqrt{3^2 + 15^2}} r \cdot \frac{15}{\sqrt{3^2 + 15^2}} r = \frac{5}{26} r^2.$$

Thus,

$$[PQRS] = 2 \cdot \frac{1}{2} r^2 + 2 \cdot \frac{5}{26} r^2 = \frac{18}{13} r^2 = \frac{N_{20}^2}{4}.$$

Since $N_{20} = 12$, our answer is $\boxed{36}$.

22. Let $d_n(x)$ be the $n$th decimal digit (after the decimal point) of $x$. For example, $d_3(\pi) = 1$ because $\pi = 3.14\underline{1}5\ldots$. For a positive integer $k$, let $f(k) = p_k^4$, where $p_k$ is the $k$th prime number. Compute the value of $\sum_{i=1}^{2023} d_{f(i)} \left(\frac{1}{1275}\right)$.

**Answer: 18199**

**Solution:** First, we want to show that

$$d_n\left(\frac{h}{k}\right) = \left\lfloor \frac{10^n h \pmod{10k}}{k} \right\rfloor$$

for a fraction $\frac{h}{k}$ with a non-terminating decimal expansion. We will analyze the long-division algorithm for computing decimal digits to show this. Let $x_n = 0.d_1 d_2 \cdots d_n$ be the first $n$ decimal digits of $\frac{h}{k}$ so that $kx_n \leq h$, which allows us to define the remainder $r_n = \frac{h}{k} - x_n$. We see that in particular $0 < 10^n(h - kx_n) < k$, as otherwise we could add $10^{-n}$ to $x_n$, incrementing $d_n$ by 1 and decreasing $10^n(h - kx_n)$ by $k$. This contradicts the fact that $x_n$ are the first $n$ digits of $\frac{h}{k}$ after the decimal point. (We also cannot have equality as otherwise the decimal would have terminated.) Since the non-terminating decimal representation of a number is unique, this changing of $d_n$ is not allowed. Therefore, we take $r_n = 10^n h \pmod{k}$, as $10^n kx_n$ is a multiple of $k$ such that $0 < 10^n h - 10^n kx_n < k$. To compute the next digit, we would multiply the remainder by 10 and subtract the largest possible multiple of $k$ from it. This looks like $d_{n+1} = \left\lfloor \frac{10 \cdot r_n}{k} \right\rfloor$, which gives the formula we wanted.

Now, the important part is determining what the period of this sequence is. This requires computing the order of 10 modulo 12750. We factor: $12750 = 10(1275) = 10(25)(51) = 2 \cdot 3 \cdot 5^3 \cdot 17$. The order of 10 modulo 12750 is then just the LCM of the orders mod each prime power. We can ignore $2, 5^3$ since they both divide $10^3$ and 3 is less than any prime to the fourth power. So we just need the order of 10 modulo 3 and 17; since $10 \equiv 1 \pmod{3}$ that order is 1, and we compute $10^2 = 100 \equiv -2 \pmod{17}$, and $(-2)^4 \equiv -1 \pmod{17}$ meaning that the order of 10 is exactly $17 - 1 = 16$. So the overall period of the sequence is 16 (after the first 3 terms where we are eliminating the powers of 2 and 5).

It remains to compute $p_n^4 \pmod{16}$. We see that $2^4 \equiv 0 \pmod{16}$, and then all other primes are odd so their square is $1 \pmod 8$ (by computation). We obtain that $(8k+1)^2 = 64k^2 + 16k + 1$ is equivalent to $1 \pmod{16}$ always. Therefore the sum we want to compute is equal to $d_{16}(\frac{1}{1275}) + 2022d_{17}(\frac{1}{1275})$ (we can't use $d_0$ and $d_1$ as we need to wait for the sequence to stabilize by becoming equivalent to 0 modulo the powers of 2 and 5.)

So, we have to compute what $r_{15}, r_{16}$ are, which we can do via CRT and repeated squaring. We know what $r_n$ will be modulo 2, 3, and $5^3$ so we only need to compute what it will be modulo 17 to get the value. We'll need to compute $10^{15}, 10^{16} \pmod{17}$. We've already shown that $10^{16} \equiv 1 \pmod{17}$, so $10^{15} \equiv 10^{-1} = 12 \pmod{17}$. Therefore $r_{15}, r_{16} \equiv 0 \pmod{250}$, $r_{15} \equiv 2(17) + 12 = 46 \pmod{51}$ and $r_{16} \equiv 1 \pmod{51}$. We see that $250 \equiv -5 \pmod{51}$, so we can quickly obtain that $r_{15} = 250$ and $r_{16} = 10(250) = 2500 \equiv 1225 \pmod{1275}$. Finally, we compute

$$d_{16} = \left\lfloor \frac{10 \cdot r_{15}}{1275} \right\rfloor = \left\lfloor \frac{2500}{1275} \right\rfloor = 1$$

and

$$d_{17} = \left\lfloor \frac{10 \cdot r_{16}}{1275} \right\rfloor = \left\lfloor \frac{12250}{1275} \right\rfloor = 9.$$

So, our final answer is $1 + 2022(9) = \boxed{18199}$.

23. A robot initially at position 0 along a number line has a *movement function* $f(u, v)$. It rolls a fair 26-sided die repeatedly, with the $k$th roll having value $r_k$. For $k \geq 2$, if $r_k > r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the positive direction. If $r_k < r_{k-1}$, it moves $f(r_k, r_{k-1})$ units in the negative direction. If $r_k = r_{k-1}$, all movement and die-rolling stops and the robot remains at its final position $x$. If $f(u, v) = (u^2 - v^2)^2 + (u-1)(v+1)$, compute the expected value of $x$.

**Answer: 225**

**Solution 1:** Let $g(u, v) = (u^2 - v^2)^2 + uv - 1$ and $h(u, v) = u - v$ be the symmetric and asymmetric parts of $f$. We will compute the robot's expected movement $C$ on the $k$th step, which is a constant independent of $k$.

$$\begin{aligned} C &= \mathbb{E}_{R_{k-1}, R_k}\left[\begin{cases} f(R_k, R_{k-1}) & \text{if } R_k > R_{k-1} \\ -f(R_k, R_{k-1}) & \text{if } R_k < R_{k-1} \\ 0 & \text{if } R_k = R_{k-1} \end{cases}\right] \\ &= \mathbb{E}_{R_{k-1}}\left[\mathbb{E}_{R_k}\left[\begin{cases} g(R_k, R_{k-1}) + h(R_k, R_{k-1}) & \text{if } R_k > R_{k-1} \\ -g(R_k, R_{k-1}) - h(R_k, R_{k-1}) & \text{if } R_k < R_{k-1} \\ 0 & \text{if } R_k = R_{k-1} \end{cases} \middle| R_{k-1}\right]\right]. \end{aligned}$$

We split this calculation into two parts. For the $g$ terms, since $R_k$ and $R_{k-1}$ are independent and identically distributed, it follows that we can swap them in the symmetric case. We can simplify as

$$\begin{aligned} &\mathbb{E}_{R_{k-1}, R_k}[g(R_k, R_{k-1}) \mid R_k > R_{k-1}]\, \mathbb{P}(R_k > R_{k-1}) \\ &\quad - \mathbb{E}_{R_{k-1}, R_k}[g(R_k, R_{k-1}) \mid R_k < R_{k-1}]\, \mathbb{P}(R_k < R_{k-1}) \\ &= \mathbb{E}_{R_{k-1}, R_k}[g(R_k, R_{k-1}) \mid R_k < R_{k-1}]\, \mathbb{P}(R_k < R_{k-1}) \\ &\quad - \mathbb{E}_{R_{k-1}, R_k}[g(R_k, R_{k-1}) \mid R_k > R_{k-1}]\, \mathbb{P}(R_k > R_{k-1}). \end{aligned}$$

The two terms of the expression are negations of each other, so this must equal 0.

For the $h$ terms, we compute

$$\begin{aligned}&\mathbb{E}_{R_{k-1},R_k}\left[R_k - R_{k-1}\,|\,R_k > R_{k-1}\right]\mathbb{P}(R_k > R_{k-1})\\ &\quad - \mathbb{E}_{R_{k-1},R_k}\left[R_k - R_{k-1}\,|\,R_k < R_{k-1}\right]\mathbb{P}(R_k < R_{k-1})\\ &= \frac{1}{N^2}\sum_{r_{k-1}=1}^{N}\left(\sum_{r_k=1}^{r_{k-1}}(r_{k-1}-r_k) + \sum_{r_k=r_{k-1}+1}^{N}(r_k - r_{k-1})\right)\\ &= \frac{1}{N^2}\sum_{r_{k-1}=1}^{N}\left(\binom{r_{k-1}}{2} + \binom{N+1-r_{k-1}}{2}\right)\\ &= \frac{1}{N^2}\cdot 2\binom{N+1}{3},\end{aligned}$$

where $N = 26$ and the last line follows from the Hockey Stick Identity. The number of times the robot rolls the die is distributed as $T \sim \text{Geometric}(\frac{1}{N})$, so by linearity of expectation

$$\mathbb{E}[X] = C\mathbb{E}[T] = \frac{(N-1)(N+1)}{3N}\cdot N = \boxed{225}.$$

**Solution 2:** By linearity of expectation and some symmetry, we can remove most of the complicated parts of $f(x, y)$, since in expectation symmetric functions in $x, y$ will sum to 0 expected movement.

Subtracting off all symmetric parts gives $g(r_k, r_{k-1}) = r_k - r_{k-1}$, and then by antisymmetry the expected value of the sums will be equal to $2r_k$.

So we can replace $f(x, y)$ with $2x$. Let $n = 26$ be the number of sides of the die for convenience. Then given that $r_2 = k$, we expect to add

$$E_k = 2\cdot\frac{1}{n}\left((k+1+\cdots+n) - (1+\cdots+k-1)\right) = \frac{1}{n}\cdot\left(\binom{n+1}{2} - \binom{k+1}{2} - \binom{k}{2}\right)$$

to our $x$ coordinate. Therefore our expected total movement is

$$\frac{1}{n}\sum_{i=1}^{n}E_i = 2\cdot\frac{1}{n^2}\left(n\binom{n+1}{2} - \binom{n+2}{3} - \binom{n+1}{3}\right) = 2\cdot\frac{1}{n^2}\left(\binom{n+1}{3}\right)$$

by use of the Hockey Stick Identity. Then since we have probability $\frac{1}{n}$ of stoppping on any given turn, we expect to move $n$ times so the expected value of the $x$ coordinate is therefore $\frac{2}{n}\binom{n+1}{3}$. Plugging in $n = 26$ gives $\frac{2(27)(26)(25)}{6(26)} = (9)(25) = \boxed{225}$.

24. Define the sequence $s_0, s_1, s_2, \ldots$ by $s_0 = 0$ and $s_n = 3s_{n-1} + 2$ for $n \geq 1$. The monic polynomial $f(x)$ defined as

$$f(x) = \frac{1}{s_{2023}}\sum_{k=0}^{32} s_{2023+k}x^{32-k}$$

can be factored uniquely (up to permutation) as the product of 16 monic quadratic polynomials $p_1, p_2, \cdots, p_{16}$ with real coefficients, where $p_i(x) = x^2 + a_i x + b_i$ for $1 \leq i \leq 16$. Compute the integer $N$ that minimizes $\left|N - \sum_{k=1}^{16}(a_k + b_k)\right|$.

**Answer: 141**

**Solution:** Our goal is to round the sum of the $a_i$ and $b_i$ to the nearest integer. The first step is to determine the recurrence relation $s_n = 3^n - 1$, which can be shown via induction. From there, the sum of the $a_i$ can easily be obtained by Vieta's formulae: the sum is equal to the negative of the coefficient of $x^{31}$, which is exactly $\frac{3^{2024}-1}{3^{2023}-1} = 3 + \frac{2}{3^{2023}-1}$.

Then, to compute the sum of the $b_i$, we bound the magnitude of the roots of $f(x)$. Each $b_i$ is equal to the product of the magnitudes of two of the roots of $f(x)$, so sufficient bounding on the magnitudes of all of the roots will net an approximation of the sum of the $b_i$ that is sufficient to get our answer. In particular, the motivation for our bounding comes from the fact that $f(x)$ is almost exactly equal to $g(x) = \frac{x^{33}-3^{33}}{x-3}$, which has roots equal to $3\omega^k, 1 \leq k \leq 32$, where $\omega = e^{\frac{2\pi i}{33}}$ is a 33rd root of unity. So, our goal will be to bound the magnitude of the roots $|x|$ of $f(x)$ by $3 - a \leq |x| \leq 3 + b$ where $a$ and $b$ are as small as possible.

In particular, we find $a$ and $b$, both less than $\frac{1}{64}$, such that $|f(x)| > 0$ whenever it's not the case that $3 - a \leq |x| \leq 3 + b$. We manipulate the given setup for something easier to work with (taking advantage of the triangle inequality):

$$\begin{aligned}
0 &< |f(x)| \\
0 &< \left| \frac{1}{3^{2023}-1} \sum_{k=0}^{32} (3^{2023+k} - 1) x^{32-k} \right| \\
0 &< \left| \sum_{k=0}^{32} (3^{2023+k} - 1) x^{32-k} \right| \\
0 &< \left| \sum_{k=0}^{32} 3^{2023+k} x^{32-k} \right| - \left| \sum_{k=0}^{32} x^{32-k} \right| \leq \left| \sum_{k=0}^{32} (3^{2023+k} - 1) x^{32-k} \right| \\
0 &< \left| \sum_{k=0}^{32} 3^{2023+k} x^{32-k} \right| - \left| \sum_{k=0}^{32} x^{32-k} \right| \leq (3^{2023} - 1)|f(x)| \\
0 &< 3^{2023} \left| \frac{x^{33} - 3^{33}}{x - 3} \right| - \left| \frac{x^{33} - 1}{x - 1} \right| \leq (3^{2023} - 1)|f(x)|
\end{aligned}$$

Therefore, we want to show sufficient conditions for

$$0 < 3^{2023} \left| \frac{x^{33} - 3^{33}}{x - 3} \right| - \left| \frac{x^{33} - 1}{x - 1} \right|,$$

and we obtain a bound of the magnitude of the roots of $f(x)$. We continue:

$$\begin{aligned}
0 &< 3^{2023} \left| \frac{x^{33} - 3^{33}}{x - 3} \right| - \left| \frac{x^{33} - 1}{x - 1} \right| \\
\left| \frac{x^{33} - 1}{x - 1} \right| &< 3^{2023} \left| \frac{x^{33} - 3^{33}}{x - 3} \right| \\
\frac{1}{3^{2023}} &< \frac{|x^{33} - 3^{33}||x - 1|}{|x^{33} - 1||x - 3|}
\end{aligned}$$

We will use this inequality to find upper and lower bounds for the magnitude $|x|$ of the roots separately. We proceed with finding the upper bound by assuming $|x| > 3$ and simplifying by extensive use of the triangle inequality:

$$\frac{1}{3^{2023}} < \frac{|x^{33} - 3^{33}||x-1|}{|x^{33}-1||x-3|}$$
$$\frac{1}{3^{2023}} < \frac{(|x|^{33} - 3^{33})(|x|-1)}{(|x|^{33}+1)(|x|+3)} \leq \frac{|x^{33} - 3^{33}||x-1|}{|x^{33}-1||x-3|}$$

Now, plugging in $|x| = \sqrt[33]{3^{33}+1}$, we get

$$\frac{(|x|^{33} - 3^{33})(|x|-1)}{(|x|^{33}+1)(|x|+3)} = \frac{(|x|-1)}{(3^{33}+2)(|x|+3)} > \frac{2}{(3^{33}+2)\cdot 7} > \frac{1}{3^{2023}}$$

We see that increasing $|x|$ will increase the value of the LHS, and in general as $|x|$ goes to infinity the LHS goes to 1. Therefore, we can say that in order for $|f(x)| = 0$, we must have $|x| < \sqrt[33]{3^{33}+1}$.

We would like to do something similar for the lower bound, but there are concerns about the case where $|x|$ is close to 1, which would give the magnitude of the fraction as close to 0 by the $(|x|-1)$ factor. So, we use our upper bound to get a decent lower bound on $|x|$, and then improve it afterwards. To demonstrate that $|x|$ is not close to 1, we use the fact that the product of the $b_i$ is equal to the product of the roots of $f(x)$, by Vieta's formulae. We have that the product of the $b_i$ is equal to $\frac{3^{2023+32}-1}{3^{2023}-1} \approx 3^{32}$. However, if we take 31 of the roots to be of the maximum magnitude $\sqrt[33]{3^{33}+1}$, then the minimum possible magnitude of any given root is almost exactly $\frac{3^{32}}{(3^{33}+1)^{\frac{31}{33}}} > 2.9999999$ (of course, it is not necessary to calculate this explicitly but it is fairly intuitive that $\sqrt[33]{3^{33}+1}$ is so close to 3 that it is impossible for this to be close to 1). This bound is actually sufficient to claim that the $b_i$ are sufficiently close to $3^2 = 9$ to finish the problem, but it may be difficult to intuit this without performing the calculations (which are certainly infeasible). However, we can easily assume that $|x| > 2$, which will allow us to improve our bound in the manner we used for the upper bound.

Now, in pursuit of a better lower bound we assume $2 < |x| < 3$ and strive to find $c$ such that $|x| < c$ implies the inequality above. We proceed by simplifying the inequality until it is in terms of $|x|$:

$$\frac{1}{3^{2023}} < \frac{|x^{33} - 3^{33}||x-1|}{|x^{33}-1||x-3|}$$
$$\frac{1}{3^{2023}} < \frac{(3^{33} - |x|^{33})(|x|-1)}{(|x|^{33}+1)(|x|+3)} \leq \frac{|x^{33} - 3^{33}||x-1|}{|x^{33}-1||x-3|}$$

by extensive use of the triangle inequality and the fact that $|x| < 3$. By plugging in $|x| = \sqrt[33]{3^{33}-1}$, we get

$$\frac{(3^{33} - |x|^{33})(|x|-1)}{(|x|^{33}+1)(|x|+3)} = \frac{1(|x|-1)}{3^{33}(|x|+3)} > \frac{1}{3^{33}\cdot 7} > \frac{1}{3^{2023}}$$

We see that decreasing $|x|$ further will increase the value of the fraction and in general this fraction gets much larger than $\frac{1}{3^{2023}}$ as $|x|$ approaches 2 (on the order of $\frac{3^{33}}{2^{33}}$), and so we take our lower bound for $|x|$ as $|x| > \sqrt[33]{3^{33}-1}$.

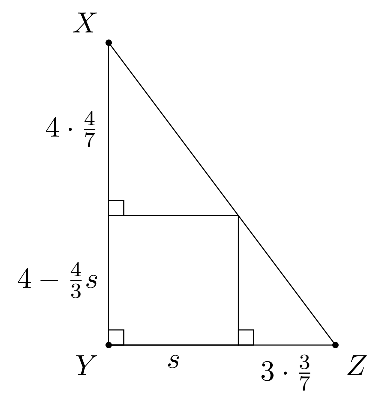
Finally, we can bound the magnitude of our roots as $\sqrt[33]{3^{33}-1} < |x| < \sqrt[33]{3^{33}+1}$, which means essentially that our $b_i$ will be between $(3^{33}-1)^{\frac{2}{33}}$ and $(3^{33}+1)^{\frac{2}{33}}$. These are sufficiently close to 9 that we may simply assume that all of the $b_i$ are equal to 9 (and this will not cause any loss of precision when rounding), and thus the nearest integer to $\sum_{k=1}^{16} a_k + b_k$ is $-3 + 9(16) = \boxed{141}$

25. Let triangle $\triangle ABC$ have side lengths $AB = 6$, $BC = 8$, and $CA = 10$. Let $S_1$ be the largest square fitting inside of $\triangle ABC$ (sharing points on edges is allowed). Then, for $i \geq 2$, let $S_i$ be the largest square that fits inside of $\triangle ABC$ while remaining outside of all other squares $S_1, \cdots, S_{i-1}$ (with ties broken arbitrarily). For all $i \geq 1$, let $m_i$ be the side length of $S_i$ and let $S$ be the set of all $m_i$. Let $x$ be the 2023rd largest value in $S$. Compute $\log_2\left(\frac{1}{x}\right)$.

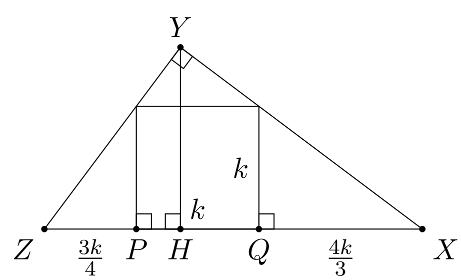
Submit your answer as a decimal $E$ to at most 3 decimal places. If the correct answer is $A$, your score for this question will be $\max(0, 25 - 2|A - E|)$, rounded to the nearest integer.

**Answer: 60.41144149627205**

**Solution:**




First, we want to determine the largest square fitting in a 3-4-5 right triangle $\triangle XYZ$ with $XY = 3, YZ = 4, XZ = 5$ (and then scale appropriately). We test two possible configurations of the square: with sides on the legs of the triangle, or with a side on the hypotenuse of the triangle. Let $s$ be the side length of this maximal square in each case. With sides on the base, we get an equation in terms of $s$: $s = 4 - \frac{4s}{3}$, giving $s = \frac{12}{7}$.




On the other hand, we can also check the largest square that has a side on the hypotenuse. Dropping an altitude from $Y$ to $\overline{XZ}$ at point $H$ gives the maximal height of such a square as $\frac{12}{5}$. Then we have two right triangles $\triangle ZHY \sim \triangle YHX \sim \triangle XYZ$. We want to draw a segment $\overline{PQ}$ on $\overline{XZ}$ containing $H$ such that the height from $P$ and $Q$ to the sides $\overline{YZ}$ and $\overline{XY}$ are both equal to $PQ$. Let the height from $P$ intersect $YZ$ at $P'$ and the height from $Q$ intersect $XY$ at $Q'$. If we set the side length of this square to $k$, then $PZ = \frac{3k}{4}$ and $QX = \frac{4k}{3}$. This follows from the fact that $\triangle PZP' \sim \triangle YZX$ and $\triangle QXQ' \sim \triangle YXZ$, and then taking advantage of the $\frac{3}{4}$ ratio of legs that gives. We then have that $ZP + PQ + QX = ZX$, or $\frac{3k}{4} + k + \frac{4k}{3} = 5$. This simplifies to $\frac{37k}{12} = 5 \implies k = \frac{60}{37}$, telling us the largest square with base on the hypotenuse has side length $\frac{60}{37}$. However, $\frac{60}{37} < \frac{12}{7}$ so we see that the largest square fitting inside $\triangle XYZ$ has side length $\frac{12}{7}$.

This gives that $s_1 = 2 \cdot \frac{12}{7} = \frac{24}{7}$. Placing this largest square inside the right triangle gives another two right triangles scaled by $\frac{3}{7}$ and $\frac{4}{7}$ respectively (again by similar triangles). Now, we have to balance out all of the multiplicative scaling and estimate which answer we will get in the end. From the 3-4-5 right triangle, placing the maximal square of side length $s$ gives two right triangles, one with maximal square side length $\frac{3s}{7}$ and one with maximal square side length $\frac{4s}{7}$. Therefore, the values of side lengths look like $s_1 \cdot \left(\frac{3}{7}\right)^m \left(\frac{4}{7}\right)^n$ for some nonnegative integers $m, n$. Then, we want to find the 2023rd largest element in this set.

We can ignore $s_1$ for now and multiply by it later. We essentially try to find some constant $K$ such that $K \leq \left(\frac{3}{7}\right)^m \left(\frac{4}{7}\right)^n$ for exactly 2023 pairs of nonnegative integers $m, n$. Taking logarithms (base 2) gives the equation

$$K_1 \leq m(\log_2 3 - \log_2 7) + n(\log_2 4 - \log_2 7)$$

for some constant $K_1 = \log_2 K < 0$. We multiply by $-1$ to obtain

$$K_2 \geq m(\log 7 - \log 3) + n(\log 7 - \log 4)$$

with $K_2 = -K_1$. Note that $K_2 = \log_2 \frac{1}{xs_1}$, so once we compute $K_2$ we can add $\log_2 s_1$ to get our final answer. Note that our current inequality describes a line in the $mn$ plane where we want to find the lattice points that are under or on the line. In order to estimate this, we need to compute estimations of $\log_2 7, \log_2 3, \log_2 4$.

Rough estimations of $2^3 \approx 7$, $2^3 \approx 3^2$, and $2^2 = 4$, give that $\log_2 7 \approx 3, \log_2 3 \approx \frac{3}{2}, \log_2 4 = 2$, which can potentially be improved by tweaking $\log_2 7$ down and $\log_2 3$ up by a small amount (say, 0.1 or 0.2). Better estimations can be obtained by using these estimations; simply raise both sides of the equation to the same power and then make small adjustments when they make improvements. For example, our original estimate gives $2^{12} = 4096 \approx 2401 = 7^4$ by raising both sides to the 4th power, but then a division by 2 on the left improves the approximation greatly. This gives $\log_2 7 \approx \frac{11}{4}$. Similarly, we can improve $\log_2 3$: we first cube both sides to get $2^9 = 512 \approx 3^6 = 729$; dividing by 2 on the left and 3 on the right gives $2^8 = 256 \approx 3^5 = 243$ which gives $\log_2 3 \approx 1.6$. If further improvements are desired, note that we now have an underestimation of $\log_2 7$ and overestimation of $\log_2 3$, so the tweaks should be in the opposite direction as before. We will adjust to $\log_2 7 \approx 2.8$ (noticing that $2^{14} = 16384 \approx 7^5 = 16807$ using the same strategy described above) and leave $\log_2 3$ untouched.

We then get the two approximations of coefficients $\log_2 7 - \log_2 3 \approx 1.2$ and $\log_2 7 - \log_2 4 \approx 0.8$. This lets us get the equation $K_2 \geq 1.2m + 0.8n$, and we want to pick $K_2$ so there are 2023 solutions to this equation. We can write the equation of the line as $n = -1.5m + K_3$ (with $K_3 = 1.25K_2$), which has intercepts $(0, K_3)$ and $(\frac{2}{3}K_3, 0)$. We want to calculate the number of lattice points inside, we can either estimate with the area of the triangle or count them more carefully. We can count the points on the boundary and use Pick's theorem to compute the exact number of lattice points (assuming integer $K_3$). We can count roughly $K_3 + \frac{2}{3}K_3 + \frac{1}{3}K_3 = 2K_3$ lattice points on the boundary, and then since the area $\frac{1}{3}K_2^2 = I + \frac{B}{2} + 1$ we get that there are $\frac{1}{3}K_3^2 + K_3 - 1$ lattice points inside the triangle. We want this to equal 2023, and we can estimate by the following manipulation:

$$\frac{1}{3}K_3^2 + K_3 - 1 = \frac{1}{3}\left(K_3 + \frac{3}{2}\right)^2 - \frac{7}{4} = 2023$$

which gives $K_3 \approx \sqrt{3 \cdot (2024.75)} - \frac{3}{2} \approx \sqrt{3(45)^2} - \frac{3}{2} \approx 45 \cdot 1.75 - \frac{3}{2} = 77.25$ (where we estimate $\sqrt{3} \approx 1.75$).

We now solve backwards to get our original constant: $K_2 = 0.8K_3 \approx 61.8$ and then we just need to subtract $\log_2 s_1 = \log_2 3 + \log_2 8 - \log_2 7 = 1.8$ which gives our final answer as $\boxed{60}$ which is within 0.4 of the correct answer and earns 24 points.

Alternate solutions that employ different approximations of logarithms, estimates of number of lattice points, or a completely different method of counting the squares also exist and can also reliably achieve nearly full points. For example, if we proceed up until the $n = -1.5m + K_3$ step and then approximate the number of lattice points as the area of the triangle $(0,0), (0, K_3), (\frac{2}{3}K_3, 0)$, you will get about $\frac{1}{3}K_3^2$ lattice points. Setting this equal to 2023 gives $\frac{1}{3}K_3^2 = 2023 \approx 2025$ and $K_3 \approx (45 - \frac{1}{45})\sqrt{3} \approx 78.71$, which leaves you with $K_2 = 0.8K_3 \approx 62.97$ and $\log_2 \frac{1}{x} = K_2 - 1.8 = 61.1$, which is within 0.7 of the correct answer and also yields 24 points.

26. For positive integers $i$ and $N$, let $k_{N,i}$ be the $i$th smallest positive integer such that the polynomial $\frac{x^2}{2023} + \frac{Nx}{7} - k_{N,i}$ has integer roots. Compute the minimum positive integer $N$ satisfying the condition $\frac{k_{N,2023}}{k_{N,1000}} < 3$. Submit your answer as a positive integer $E$. If the correct answer is $A$, your score for this question will be $\max\left(0, 25\min\left(\frac{A}{E}, \frac{E}{A}\right)^{\frac{3}{2}}\right)$, rounded to the nearest integer.

**Answer: 232**

**Solution:** We first multiply our quadratic by 2023, which keeps the roots the same. We now have the expression $x^2 + 289Nx - 2023k_{N,i}$. Now, if this were to have integer solutions, because the constant term is negative, this quadratic's roots would be $289N + a$ and $-a$ for some nonnegative integer $a$. So, we have $(289N + a)a = 2023k_{N,i}$. Now, we need to ensure that $(289N + a)a$ is divisible by $7 \cdot 17^2$.

To do so, we start by analyzing this product modulo 17. In particular, because $17|289$, we have that $289N + a \equiv a \pmod{289}$, and also modulo 17. Therefore, if $a \not\equiv 0 \pmod{17}$, then $(289N + a)a \equiv a^2 \not\equiv 0 \pmod{17}$, and therefore, $(289N + a)a \not\equiv 0 \pmod{289}$. If $a \equiv 0 \pmod{17}$, then $(289N + a)$ and $a$ would both be divisible by 17, and so $(289N + a)a \equiv 0 \pmod{289}$.

Next, in order to analyze the product modulo 7, we break this into cases depending on whether $7 \mid N$:

- When $7 \mid N$, then we have that $289N \mid 7$, and so, in order for $(289N + a)a$ to be divisible by 7, we must have that $a \mid 7$, just like when we considered 17.
- When $7 \nmid N$, then $289N \not\equiv 0 \pmod{7}$. Therefore, $289N + a$ is divisible by 7 for a different $a$ residue class than $0 \pmod{7}$. Therefore, are two residue classes modulo 7 that $a$ can be.

So, in summary, when $7 \mid N$, then $a$ must be a multiple of $7 \cdot 17 = 119$, and when $7 \nmid N$, there are two residue classes modulo 119 that $a$ could be. Thus,

$$2023k_{N,i} \approx \begin{cases} 119i \cdot (119i + 289N) & \text{if } 7 \mid N \\ \frac{119i}{2}\left(\frac{119i}{2} + 289N\right) & \text{if } 7 \nmid N \end{cases}.$$

and

$$k_{N,i} \approx \begin{cases} \frac{1}{2023} \cdot 119i \cdot (119i + 289N) & \text{if } 7 \mid N \\ \frac{1}{2023} \cdot \frac{119i}{2}\left(\frac{119i}{2} + 289N\right) & \text{if } 7 \nmid N \end{cases}.$$

We want to determine the minimum $N$ such that $\frac{k_{N,2023}}{k_{N,1000}} < 3$. We can also break this up into cases:

- When $7 \mid N$, then we are considering the inequality

$$\begin{aligned}\frac{(119)(2023)\cdot((119)(2023)+289N)}{(119)(1000)\cdot((119)(1000)+289N)} &< 3\\ \frac{(119)(2023)+289N}{(119)(1000)+289N} &< \frac{3000}{2023}\\ \frac{(119)(1023)}{(119)(1000)+289N} &< \frac{977}{2023}\\ \frac{(7)(1023)}{(7)(1000)+17N} &< \frac{977}{2023}.\end{aligned}$$

  We can approximate this to $\frac{(7)(1023)}{(7)(1000)+17N} < \frac{1}{2}$, so $17N \approx \frac{(7)(1046)}{17} = 430$.

- When $7 \mid N$, then we are considering the inequality

$$\begin{aligned}\frac{(119)\left(\frac{2023}{2}\right)\cdot((119)\left(\frac{2023}{2}\right)+289N)}{(119)\left(\frac{1000}{2}\right)\cdot((119)\left(\frac{1000}{2}\right)+289N)} &< 3\\ \frac{(119)\left(\frac{2023}{2}\right)+289N}{(119)\left(\frac{1000}{2}\right)+289N} &< \frac{3000}{2023}\\ \frac{(119)\left(\frac{1023}{2}\right)}{(119)\left(\frac{1000}{2}\right)+289N} &< \frac{977}{2023}\\ \frac{(7)\left(\frac{1023}{2}\right)}{(7)\left(\frac{1000}{2}\right)+17N} &< \frac{977}{2023}\\ \frac{(7)\,(1023)}{(7)\,(1000)+34N} &< \frac{977}{2023}\end{aligned}$$

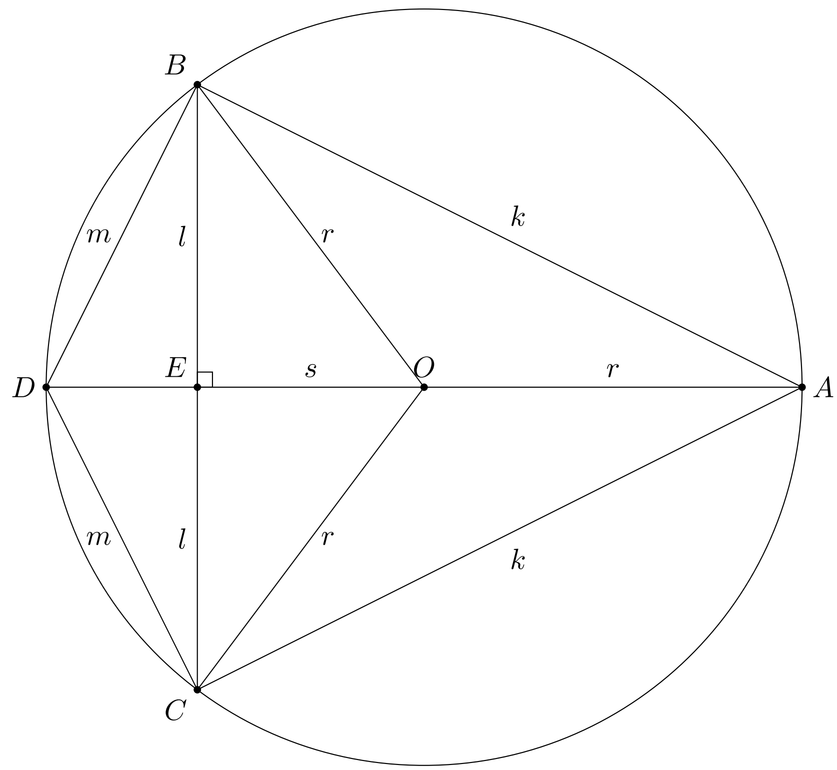
  Again, we can approximate this to $\frac{(7)(1023)}{(7)(1000)+34N} < \frac{1}{2}$, so $N \approx \frac{(7)(1046)}{34} = 215$.

Since $215 < 430$, we achieve an estimate of $\boxed{215}$, which gives us 22 points. Without approximating $\frac{977}{2023}$, we can even get slightly more points. Also note that if we approximated $\frac{(7)(1046)}{34} \approx \frac{(7)(1000)}{\frac{100}{3}} = 210$, we would still barely achieve 22 points.

27. Let $\omega$ be a circle with positive integer radius $r$. Suppose that it is possible to draw isosceles triangle with integer side lengths inscribed in $\omega$. Compute the number of possible values of $r$ where $1 \le r \le 2023^2$. Submit your answer as a positive integer $E$. If the correct answer is $A$, your score for this question will be $\max\left(0, 25\left(3-2\max\left(\frac{A}{E},\frac{E}{A}\right)\right)\right)$, rounded to the nearest integer.

**Answer: 217602**

**Solution:**




Let $N = 2023^2$ for convenience. We essentially need that for some integers $s, k$, and $m$ that either $(r+s)^2 + r^2 - s^2 = k^2$ or $(r-s)^2 + r^2 - s^2 = m^2$, and we also need $r^2 - s^2 = l^2$ for an integer $l$ in both cases. We begin with the first case: then, we have the Pythagorean triple $r+s, l, k$. However, $r^2 - s^2 = l^2$ so that $s, l, r$ is another Pythagorean triple. Then, to generate primitive Pythagorean triples for $s, l, r$, we set $s = a^2 - b^2, l = 2ab, r = a^2 + b^2$ for integers $a, b$ where we need $a$ and $b$ to be coprime and exactly one of $a$ and $b$ to be even. However, we need to make sure $r+s, l, k$ is also a Pythagorean triple; we have that $k^2 = 2r(r+s)$ by simplification, and we know that $r+s = 2a^2$ means that $k^2 = 4a^2r$. Therefore, $r$ must also be a perfect square, i.e. $r = c^2$, which means that we have another Pythagorean triple $a^2 + b^2 = c^2$ and this means we must parameterize $a, b, c$ like we did for $s, l, r$, getting $a = x^2 - y^2, b = 2xy, c = x^2 + y^2$.

Now, we need to check the other case where we are instead requiring $(r-s)^2 + r^2 - s^2 = m^2$. It turns out that if $k$ is an integer, then $m$ is an integer: notice that by the same parameterization from the first case, we have $(r-s)^2 + r^2 - s^2 = 2r(r-s) = m^2 = 2r(2b^2) = m^2$, which is again true if and only if $r$ is a perfect square. But we know that for $k$ to be an integer, we need $r$ to be a perfect square, which means either both $m$ and $k$ are integers or neither are. So, our characterization of all possible $r$ is given by $\lambda p^2$, where $\lambda$ is some positive integer and $p$ is some prime expressible as the sum of two squares (which just means $1 \pmod 4$). We obtain this from scaling up any of our initial primitive Pythagorean triple solutions where $r$ is a perfect square which is also the hypotenuse of an integer right triangle.

Let $P$ be the set of all valid $p$. We then need to compute the sum

$$\sum_{p \in P} \left\lfloor \frac{N}{p^2} \right\rfloor .$$

We can estimate this sum by estimating $\sum_{p \in P} \frac{1}{p^2}$ and multiplying by $N$. We estimate this constant coefficient by adding up a few terms:

$$\frac{1}{25} + \frac{1}{169} + \frac{1}{289} + \frac{1}{841} \approx 0.04 + 0.006 + 0.0035 + 0.0011 = 0.0506.$$

We then multiply by $N$ to get the approximation $0.0506(2023^2) = 0.0511(2000^2 + 2(23)(2000) + 23^2) \approx 202500 + 4600 + 26 = \boxed{207126}$, which is within 10,000 of the real answer, 217602, and gives you a score of 22. Adding a few more terms more meticulously (or guessing a good value for the coefficient like $\frac{1}{19}$) can get you a perfect score of 25. Alternatively, you can guess that the constant coefficient is something like $\frac{1}{20}$ and that $2023^2 \approx 4 \cdot 10^6$ and get the answer 200000, which is also a fairly good estimate and gets a score of 20.

If you don't find the parameterization of $r$, but somehow stumble into the $r = 25$ solution and scale appropriately, you can multiply $\frac{1}{25}$ into $N$ and get $\approx 163000$, which gives about 10 points.