1. Arjun eats twice as many chocolates as Theo, and Wen eats twice as many chocolates as Arjun. If Arjun eats 6 chocolates, compute the total number of chocolates that Arjun, Theo, and Wen eat.

   Answer: 21

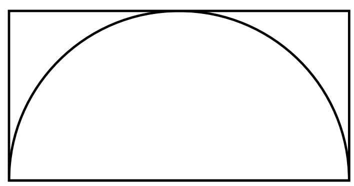
   Solution: Since Arjun ate 6 chocolates, then Theo ate $\frac{6}{2} = 3$ chocolates and Wen ate $2x = 12$ chocolates. In total, the three eat $6 + 3 + 12 = \boxed{21}$ chocolates.

2. Compute $1 \times 4 - 2 \times 3 + 2 \times 5 - 3 \times 4 + 3 \times 6 - 4 \times 5 + 4 \times 7 - 5 \times 6 + 5 \times 8 - 6 \times 7$.

   Answer: -10

   Solution: Note that when we group every pair of terms together we get the following: $(1 \times 4 - 2 \times 3) + (2 \times 5 - 3 \times 4) + (3 \times 6 - 4 \times 5) + (4 \times 7 - 5 \times 6) + (5 \times 8 - 6 \times 7) + (6 \times 9 - 7 \times 8) = (4 - 6) + (10 - 12) + (18 - 20) + (28 - 30) + (40 - 42) = 5 \cdot (-2) = \boxed{-10}$.

3. A semicircle of radius 2 is inscribed inside of a rectangle, as shown in the diagram below. The diameter of the semicircle coincides with the bottom side of the rectangle, and the semicircle is tangent to the rectangle at all points of intersection. Compute the length of the diagonal of the rectangle.



   Answer: $2\sqrt{5}$

   Solution: Two sides of the rectangle are length 4 as they coincide with the diameter of the circle. The other two sides are length 2 because they are the same length as the radius of the circle. Thus, by the Pythagorean Theorem, the length of the diagonal is $\sqrt{2^2 + 4^2} = \boxed{2\sqrt{5}}$.

4. Suppose a, b, and c are numbers satisfying the three equations:

$$a + 2b = 20,$$
$$b + 2c = 2,$$
$$c + 2a = 3.$$

   Find $9a + 9b + 9c$.

   Answer: 75

   Solution: If we sum the three equations, we get:

$$3a + 3b + 3c = 25.$$

   Then, multiplying both sides by 3 gives us $9a + 9b + 9c = 3 \cdot 25 = \boxed{75}$.

5. Lakshay chooses two numbers, m and n, and draws two lines, $y = mx + 3$ and $y = nx + 23$. Given that the two lines intersect at $(20, 23)$, compute $m + n$.

   Answer: 1

   Solution: Since the point $(20, 23)$ lies on both lines, we can plug it into our two equations to get $23 = 20m + 3$ and $23 = 20n + 23$. Solving these equations, we get that $m = 1$ and $n = 0$ for a final answer of $1 + 0 = \boxed{1}$.

6. Compute the three-digit number that satisfies the following properties:

   - The hundreds digit and ones digit are the same, but the tens digit is different.
   - The number is divisible by 9.
   - When the number is divided by 5, the remainder is 1.

   Answer: 171

   Solution: For our number to be one more than a multiple of 5, it must end in a 1 or a 6. Since the hundreds digit is the same as the ones digit, the number is either of the form $\underline{1}\,\underline{A}\,\underline{1}$ or $\underline{6}\,\underline{B}\,\underline{6}$. The divisibility rule for 9 gives that $1 + A + 1$ and $6 + B + 6$ would have to be multiples of 9, so $A = 7$ and $B = 6$. This gives 171 and 666 as candidate numbers, but the tens digit must be different from the other digits, so the only three-digit number satisfying all conditions is $\boxed{171}$.

7. Recall that an arithmetic sequence is a sequence of numbers such that the difference between any two consecutive terms is the same. Suppose x_1, x_2, x_3 forms an arithmetic sequence. If $x_2 = 2023$, compute $x_1 + x_2 + x_3$.

   Answer: 6069

   Solution: Suppose the common difference in this arithmetic sequence is d. Then, we can write $x_1 = x_2 - d = 2023 - d$ and $x_3 = x_2 + d = 2023 + d$. That means that

   $$x_1 + x_2 + x_3 = (2023 - d) + 2023 + (2023 + d) = 3 \cdot 2023 = \boxed{6069}.$$

8. One of Landau's four unsolved problems asks whether there are infinitely many primes p such that $p - 1$ is a perfect square. How many such primes are there less than 100?

   Answer: 4

   Solution: We could check every prime number less than 100, but it is easier and faster to check every number less than 100 that is one more than a perfect square. Also, we do not need to check odd squares larger than 1 since all primes larger than 2 are odd. We have $1^2 + 1 = 2$, $2^2 + 1 = 5$, $4^2 + 1 = 17$, $6^2 + 1 = 37$, which are all prime, while $8^2 + 1 = 65$ is not prime. So, the answer is $\boxed{4}$.

9. The boxes in the expression below are filled with the numbers $3, 4, 5, 6, 7,$ and 8, so that each number is used exactly once. What is the least possible value of the expression?

   $$\square \times \square + \square \times \square - \square \times \square$$

   Answer: −18

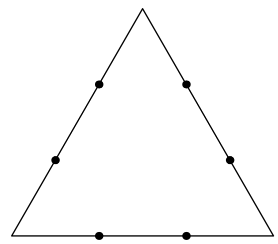
   Solution: To minimize the expression, we want the product with the minus sign to be as large as possible. Therefore, we use the largest numbers to make the final product 7×8. We then try each of the possibilities for the other two products:

   $$3 \times 4 + 5 \times 6 = 12 + 30 = 42$$
   $$3 \times 5 + 4 \times 6 = 15 + 24 = 39$$
   $$3 \times 6 + 4 \times 5 = 18 + 20 = 38.$$

   So, the smallest possible value of the expression is

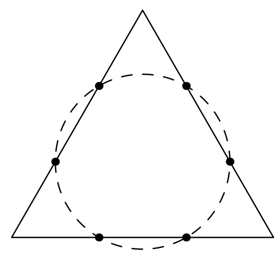
   $$3 \times 6 + 4 \times 5 - 7 \times 8 = 38 - 56 = \boxed{-18}.$$

10. Consider an equilateral triangle with side length 9. Each side is divided into 3 equal segments by 2 points, for a total of 6 points. Compute the area of the circle passing through these 6 points.



Answer: 9π

Solution:



First, we draw in the circle, as shown in the diagram. The hexagon created by the six points is regular because all of the segments have length $9/3 = 3$ and all of the equilateral triangle's angles are 60°. Furthermore, the diameter of the circle is the longest diagonal of the hexagon, and the longest diagonal of a regular hexagon is twice the length of its side length, and thus is 6. Therefore, the area of the circle must be $3^2\pi = \boxed{9\pi}$.

11. Consider two geometric sequences $16, a_1, a_2, \ldots$ and $56, b_1, b_2, \ldots$ with the same common nonzero ratio. Given that $a_{2023} = b_{2020}$, compute $b_6 - a_6$.

Answer: 490

Solution: Let r represent the common ratio of both sequences. Then, $a_{2023} = b_{2020}$ implies $16 \cdot r^{2023} = 56 \cdot r^{2020}$. Because $r \neq 0$, this means $r^3 = \frac{r^{2023}}{r^{2020}} = \frac{56}{16} = \frac{7}{2}$. Then we have $a_6 = 16 \cdot r^6 = 16 \cdot \left(r^3\right)^2 = 16 \cdot \left(\frac{7}{2}\right)^2 = 196$. Similarly, $b_6 = 56 \cdot r^6 = 56 \cdot \left(r^3\right)^2 = 56 \cdot \left(\frac{7}{2}\right)^2 = 686$. Finally, we get that $b_6 - a_6 = 686 - 196 = \boxed{490}$.

12. Find the greatest integer less than $\sqrt{10} + \sqrt{80}$.

Answer: 12

Let $x = \sqrt{10} + \sqrt{80}$ be the quantity in question.
Solution 1: We first present an intuition-based solution. Let $\sqrt{10} = 3 + a$ for some small a. Similarly, $\sqrt{80} = 9 - b$ for some small b. Then, $x = 12 + a - b$. We'd like to compare the magnitudes of a and b. Intuitively, since 9 is smaller than 81, adjusting the numbers by 1 affects their square roots differently. In particular, the difference $\sqrt{10} - \sqrt{9}$ intuitively should be greater than $\sqrt{81} - \sqrt{80}$ because $\sqrt{10}$ should probably be closer to $\sqrt{16} = \sqrt{9} + 1$ than $\sqrt{80}$ is to $\sqrt{64} = \sqrt{81} - 1$. Thus, $a - b$ is a small but positive number, and thus the answer is $\boxed{12}$.
Solution 2: A more rigorous solution is as follows. If we consider x^2, it is equal to $10 + 80 + 2\sqrt{800} = 90 + 40\sqrt{2}$. Recall that $\sqrt{2} > 1.4$ because $1.4^2 = 1.96 < 2$. So, we have $x^2 >$

$90+40\cdot 1.4 = 90+56 = 146$. Taking the square root of both sides yields $x \geq \sqrt{146} > \sqrt{144} = 12$. We can easily check that $\sqrt{10} < 4$ and $\sqrt{80} < 9$, and so $x < 13$. Therefore, the answer is $\boxed{12}$.
Solution 3: A third way to see this is to note that $(\sqrt{10}+\sqrt{9})(\sqrt{10}-\sqrt{9}) = 1 = (\sqrt{81}+\sqrt{80})(\sqrt{81}-\sqrt{80})$, and so $\sqrt{10}-\sqrt{9} = \frac{1}{\sqrt{10}+\sqrt{9}} > \frac{1}{\sqrt{81}+\sqrt{80}} = \sqrt{81}-\sqrt{80}$. In other words, $\sqrt{10}$ is farther away from $\sqrt{9} = 3$ is than $\sqrt{80}$ is from $\sqrt{81} = 9$, and thus the answer is $\boxed{12}$.
The exact value of $\sqrt{10}+\sqrt{80}$ is approximately 12.107.

13. Three people, Pranav, Sumith, and Jacklyn, are attending a fair. Every time a person enters or exits, the groundskeeper writes their name down in chronological order. If each person enters and exits the fairgrounds exactly once, in how many ways can the groundskeeper write down their names?

Answer: 90

Solution: Let A, B, C represent the events of the three friends entering and let X, Y, Z represent the events of friends exiting. Then the number we're looking for is the number of permutations of $ABCXYZ$ where A comes before X, B comes before Y, and C comes before Z. This is $\frac{6!}{2^3}$, as we take any permutation and then swap $(A, X), (B, Y)$, or (C, Z) if needed. There are 2^3 possible combinations of swaps we may need to do, and exactly one of these swaps is valid. So, for every $2^3 = 8$ permutations, there is exactly one valid permutation. Thus, the answer is $\frac{6!}{2^3} = \boxed{90}$.

14. For real numbers x and y, suppose that $|x| - |y| = 20$ and $|x| + |y| = 23$. Compute the sum of all possible distinct values of $|x - y|$.

Answer: 43

Solution: Adding the equations gives $2|x| = 43$, which means $(x, y) = \left(\pm\frac{43}{2}, \pm\frac{3}{2}\right)$. Thus, the possible values of $x - y$ are $\pm 20, \pm 23$. However, the negative solutions are extraneous because we are interested in $|x - y|$. The sum of all unique possible values therefore is $20 + 23 = \boxed{43}$.

15. Find the number of positive integers n less than 10000 such that there are more 4's in the digits of $n + 1$ than in the digits of n.

Answer: 1111

Solution: Note that increasing n by one can only affect more than the last digit if the last digit is a 9, in which case we also essentially add one to the tens digit. Thus, without carrying, the only way to increase in the number of 4s in the base-10 expansion is for the last digit to be a 3. However, if we carry once, we need the units digit to be a 9 and the tens digit to be a 3, and so on. It follows that, for a number k, there are more 4s in the base-10 representation of $k + 1$ than k if and only if k ends in $39\ldots9$, where the number of 9s is a nonnegative integer.

Now, for every $n \in \{0, 1, 2, 3\}$, suppose k has the form

$$d_n d_{n-1} \cdots d_1 3 \underbrace{9\ldots9}_{4-n-1}.$$

There are 10^n positive integers of this form, as the d_i can be any digit. Summing over all n, we the total number of positive integers is $1 + 10 + 100 + 1000 = \boxed{1111}$.

16. Let n be the smallest positive integer such that there exist integers, a, b, and c, satisfying:

$$\frac{n}{2} = a^2, \quad \frac{n}{3} = b^3, \quad \frac{n}{5} = c^5.$$

Find the number of positive integer factors of n.

Answer: 1232

Solution: We apply the Chinese Remainder Theorem to the exponents. First, it's clear that the n with the least number of divisors can be written in the form $2^x \cdot 3^y \cdot 5^z$.

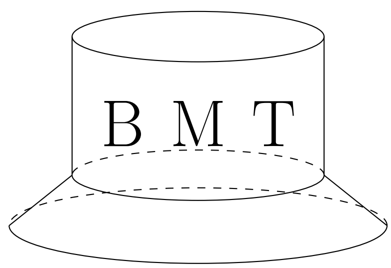
We can build equations from the representations. First, for the smallest value of x, we have that x must satisfy:

$$\begin{aligned} x &\equiv 1 \pmod 2 \\ x &\equiv 0 \pmod 3 \\ x &\equiv 0 \pmod 5. \end{aligned}$$

Solving these three equations from the Chinese Remainder Theorem gives $x = 15$ as the smallest nonnegative integral solution. Similarly, setting up equations for y and z gives us $y = 10$ and $z = 6$.

From these values, the number of factors is $(15+1)\cdot(10+1)\cdot(6+1) = \boxed{1232}$.

17. Jingyuan is designing a bucket hat for BMT merchandise. The hat has the shape of a cylinder on top of a truncated cone, as shown in the diagram below. The cylinder has radius 9 and height 12. The truncated cone has base radius 15 and height 4, and its top radius is the same as the cylinder's radius. Compute the total volume of this bucket hat.



Answer: 1560π

Solution: Recall that the volume of a cone with height h and radius r is $\frac{\pi h r^2}{3}$. The truncated cone is equal to the difference between a cone with radius 15 and height $4+x$ and a cone with radius 9 and height x for some value of x. Using similar triangles, we have that $\frac{x}{9} = \frac{x+4}{15}$. Solving gives $x = 6$. Thus, the volume of the larger cone is $\frac{(10)(15)^2\pi}{3} = 750\pi$ and the volume of the smaller cone is $\frac{(6)(9)^2\pi}{3} = 162\pi$. Thus, the volume of the truncated cone is $750\pi - 162\pi = 588\pi$. Next, the volume of the cylinder is $r^2 \cdot h \cdot \pi = 9^2 \cdot 12 \cdot \pi = 972\pi$. Thus, the total volume of the bucket hat is $588\pi + 972\pi = \boxed{1560\pi}$.

18. Kait rolls a fair 6-sided die until she rolls a 6. If she rolls a 6 on the Nth roll, she then rolls the die N more times. What is the probability that she rolls a 6 during these next N times?

Answer: $\frac{6}{11}$

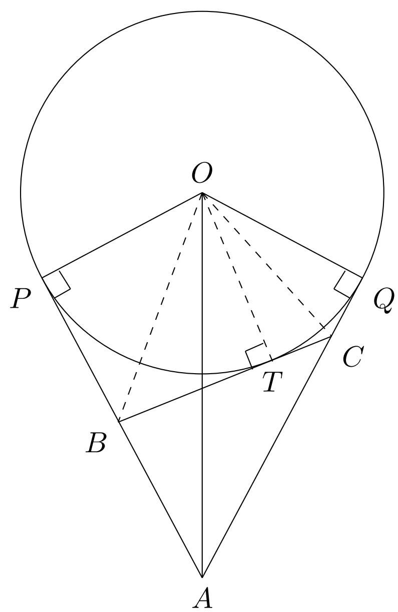
Solution: Note that, for every positive integer i, the probability that Kait first rolls a 6 on the ith roll is $\left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right)$. Then, given that she rolls a 6 on the ith roll, the probability that she doesn't roll a 6 on the next i rolls is $\left(\frac{5}{6}\right)^{i}$. Therefore, the probability that she rolls a 6 on the ith roll and then not on the next i rolls is $\left(\frac{5}{6}\right)^{2i-1}\left(\frac{1}{6}\right)$. Thus, the final probability is

$\left(\frac{5}{6}\right)^1\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\cdots$. This is a geometric sequence with initial term $\frac{5}{36}$ and common ratio of $\left(\frac{5}{6}\right)^2=\frac{25}{36}$. Therefore, the probability that she *won't* roll a 6 during these next N rolls is $\frac{\frac{5}{36}}{1-\frac{25}{36}}=\frac{5}{11}$. We want the probability that the complementary event occurs, or $1-\frac{5}{11}=\boxed{\frac{6}{11}}$.

19. Let ω be a circle with center O and radius 8, and let A be a point such that $AO=17$. Let P and Q be points on ω such that line segments $\overline{AP}$ and $\overline{AQ}$ are tangent to ω. Let B and C be points chosen on $\overline{AP}$ and $\overline{AQ}$, respectively, such that $\overline{BC}$ is also tangent to ω. Compute the perimeter of triangle $\triangle ABC$.

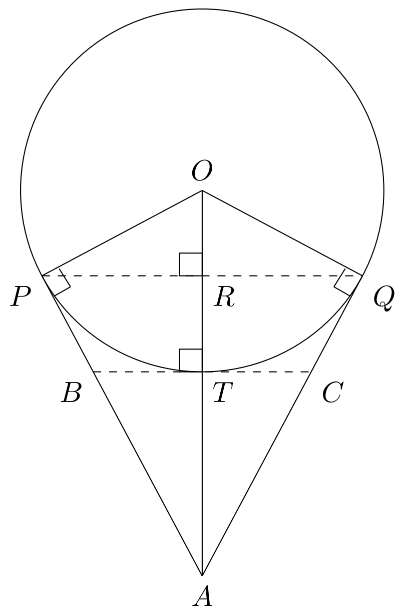
Answer: 30

Solution 1:



Let $\overline{BC}$ be tangent to ω at T. Connecting $\overline{OT}$ gives us $OP=OT=OQ=8$. By the Pythagorean Theorem, we get $QA=\sqrt{17^2-8^2}=15$. Since $OP=OT$ and $\angle OPB=\angle OTB=90^\circ$, $\triangle OBP\cong\triangle OBT$. Thus, $PB=BT$. With the same idea, we get $\triangle OCT\cong\triangle OCQ$, and from this, $CT=CQ$. Therefore, the perimeter of $\triangle ABC=AB+BC+AC=AB+BT+TC+AC=BA+BP+CQ+AC=AP+AQ=15+15=\boxed{30}$.

Solution 2:



An alternative solution can be to select $\overline{BC}$ such that $\overline{BC} \parallel \overline{PQ}$. Let $\overline{BC}$ be tangent to ω at T and $\overline{PQ}$ intersect $\overline{OA}$ at R. Since $\angle OPR + \angle POR = 90^\circ = \angle OPR + \angle APR$, $\angle POR = \angle APR$. In triangles $\triangle POR$ and $\triangle APR$, $\angle POR = \angle APR$, and $\angle ORP = \angle PRA = 90^\circ$, so $\triangle POR \sim \triangle APR$. Also, since $\angle POR = \angle AOP$ and $\angle ORP = \angle OPA = 90^\circ$, $\triangle POR \sim \triangle AOP$. This effectively means $\triangle APR \sim \triangle AOP$. Furthermore, since $\overline{BC} \parallel \overline{PQ}$, $\overline{BC} \perp \overline{AO}$, we also get $\triangle ABT \sim \triangle APR$. By the Pythagorean Theorem, $QA = \sqrt{17^2 - 8^2} = 15$. Since $\triangle APO$ has side lengths in the ratio of $8 : 15 : 17$, so does $\triangle ATB$. So, since $AT = AO - OT = 17 - 8 = 9$, we get $BT = 9 \cdot \frac{8}{15} = \frac{24}{5}$ and $AB = 9 \cdot \frac{17}{15} = \frac{51}{5}$. Therefore, the perimeter of $\triangle ABC = AB + BC + AC = AB + BT + TC + AC = 2 \cdot \left(\frac{24}{5} + \frac{51}{5}\right) = 2 \cdot 15 = \boxed{30}$. We can verify that setting $\overline{BC} \parallel \overline{PQ}$ does not violate any of the constraints of the problem.

20. Call a positive integer, n, *ready* if all positive integer divisors of n have a ones digit of either 1 or 3. Let S be the sum of all positive integer divisors of 32! that are ready. Compute the remainder when S is divided by 131.

Answer: 71

Solution: We claim that n is ready if and only if it has at most one prime factor ending in 3 (counted with multiplicity!) and the rest of the prime factors end in 1. To see that this is necessary, note that all prime factors must end in 1 or 3 by hypothesis, and if there is more than one prime factor ending in 3, then their product makes a divisor of n ending in 9. Conversely, this condition is sufficient: any divisor d of such an n either does not have the prime factor ending in 3 (which means d ends in 1) or has exactly one prime factor ending in 3 (which means d ends in 3).

It remains to compute our sum. The primes of interest are $\{11, 31\}$ which end in 1 and $\{3, 13, 23\}$ which end in 3. Powers of 11 can go up to 11^2, powers of 31 can go up to 31^1, and we may have at most one prime from $\{3, 13, 23\}$. Summing over all such prime factorizations, we are computing

$$\sum_{a=0}^{2}\sum_{b=0}^{1}\sum_{m\in\{1,3,13,23\}} 11^a \cdot 31^b \cdot m = \left(1 + 11 + 11^2\right)(1 + 31)(1 + 3 + 13 + 23) = 133 \cdot 32 \cdot 40.$$

Considering this product modulo 131, we see it is equivalent to

$$133 \cdot 32 \cdot 40 \equiv 2 \cdot 32 \cdot 40 \equiv 2 \cdot 128 \cdot 10 \equiv 2 \cdot -3 \cdot 10 \equiv \boxed{71}.$$

21. Let p, q, and r be the three roots of the polynomial $x^3 - 2x^2 + 3x - 2023$. Suppose that the polynomial $x^3 + Bx^2 + Mx + T$ has roots $p + q$, $p + r$, and $q + r$ for real numbers B, M, and T. Compute $B - M + T$.

Answer: 2006

Solution 1: Because p, q, and r are the roots of the polynomial, we have $x^3 - 2x^2 + 3x - 2023 = (x - p)(x - q)(x - r)$. By Vieta's Formulas, we have: $p + q + r = 2$, $pq + qr + pr = 3$, and $pqr = 2023$. Applying Vieta's on $x^3 + Bx^2 + Mx + T$, we have that

$$B = -((p + q) + (p + r) + (q + r)) = -2p - 2q - 2r = -2(p + q + r) = -2(2) = -4.$$

Also,

$$\begin{aligned} M &= (p + q)(q + r) + (p + q)(p + r) + (p + r)(q + r) \\ &= (2 - r)(2 - p) + (2 - r)(2 - q) + (2 - q)(2 - p) \\ &= pq + qr + pr - 4(p + q + r) + 12 \\ &= 3 - 4(2) + 12 \\ &= 7. \end{aligned}$$

Finally,

$$\begin{aligned} T &= -(p + q)(q + r)(p + r) \\ &= -(2 - r)(2 - p)(2 - q) \\ &= (r - 2)(p - 2)(q - 2) \\ &= pqr - 2(pq + qr + pr) + 4(p + q + r) - 8 \\ &= 2023 - 2(3) + 4(2) - 8 \\ &= 2017. \end{aligned}$$

Thus, $B - M + T = -4 - 7 + 2017 = \boxed{2006}$.

Solution 2: Let $f(x) = x^3 - 2x^2 + 3x - 2023$ and $g(x) = x^3 + Bx^2 + Mx + T$. We know by Vieta's that $p + q + r = 2$. Thus, the roots of $g(x)$ are $2 - p$, $2 - q$, and $2 - r$. Then

$$\begin{aligned} B - M + T &= g(-1) + 1 \\ &= (-1 - (2 - p))(-1 - (2 - q))(-1 - (2 - r)) + 1 \\ &= (-3 + p)(-3 + q)(-3 + r) + 1 \\ &= -(3 - p)(3 - q)(3 - r) + 1 \\ &= -f(3) + 1 \\ &= -((3)^3 - 2(3)^2 + 3(3) - 2023) + 1 \\ &= \boxed{2006}. \end{aligned}$$

22. Triangle $\triangle ABC$ has side lengths $AB = 8$, $BC = 15$, and $CA = 17$. Circles ω_1 and ω_2 are externally tangent to each other and within $\triangle ABC$. The radius of circle ω_2 is four times the

radius of circle ω_1. Circle ω_1 is tangent to $\overline{AB}$ and $\overline{BC}$, and circle ω_2 is tangent to $\overline{BC}$ and $\overline{CA}$. Compute the radius of circle ω_1.

Answer: $\frac{5}{7}$

Solution: Let P be the center of ω_1, Q be the center of ω_2, and the radius of the circle ω_1 be r. Divide triangle $\triangle ABC$ into triangles $\triangle AQB$, $\triangle BQC$, and $\triangle CQA$; the key idea is that the heights of these triangles are multiples of r, and we can use the areas of the triangles to find r. Since circle ω_2 is tangent to $\overline{BC}$ and $\overline{CA}$, the heights of triangles $\triangle BQC$ and $\triangle CQA$ to the sides of triangle $\triangle ABC$ are $4r$, so it remains to find the height of triangle $\triangle AQB$. Let X and Y be the points of tangency of $\overline{BC}$ to circles ω_1 and ω_2, respectively. $PXYQ$ is a trapezoid with $\angle PXY = \angle QYX = 90^\circ$, $PX = r$, $QY = 4r$, and $PQ = r + 4r = 5r$. Then, by the Pythagorean Theorem, $XY = \sqrt{PQ^2 - (QY - PX)^2} = \sqrt{(5r)^2 - (3r)^2} = 4r$. So, the height of triangle $\triangle AQB$ is $BY = BX + XY = 5r$.

The areas of the three triangles mentioned above are $\dfrac{1}{2} \cdot 8 \cdot 5r = 20r$, $\dfrac{1}{2} \cdot 15 \cdot 4r = 30r$, and $\dfrac{1}{2} \cdot 17 \cdot 4r = 34r$, and the area of triangle $\triangle ABC$ is $\frac{1}{2} \cdot 8 \cdot 15 = 60$. Therefore, $20r + 30r + 34r = 84r = 60$ and $r = \boxed{\dfrac{5}{7}}$.

23. Let N be the number of positive integers x less than $210 \cdot 2023$ such that

$$\operatorname{lcm}(\gcd(x, 1734), \gcd(x + 17, x + 1732))$$

divides 2023. Compute the sum of the prime factors of N with multiplicity. (For example, if $S = 75 = 3^1 \cdot 5^2$, then the answer is $1 \cdot 3 + 2 \cdot 5 = 13$).

Answer: 51

Solution: First, we clean up the expression slightly to make it a bit easier for us. In particular, by the Euclidean Algorithm,

$$\gcd(x + 17, x + 1732) = \gcd(x + 17, (x + 1732) - (x + 17)) = \gcd(x + 17, 1715).$$

So, our new problem involves when $\operatorname{lcm}(\gcd(x, 1734), \gcd(x + 17, 1715))$ divides 2023. In order for this to be satisfied, we simply need that $\gcd(x, 1734)$ and $\gcd(x+17, 1715)$ both divide 2023.

Now, it would be good to know what the prime factorizations of 1715 and 1734 are. Dividing 1715 by 5 leaves 343, so $1715 = 5 \cdot 7^3$. Additionally, since $34 = 2 \cdot 17$, then 17 divides 1734, and in particular $1734 = 17 \cdot 102$, which is $2 \cdot 3 \cdot 17^2$. Alternatively, we see that 2 and 3 both divide 1734, and then $\frac{1734}{6} = 289 = 17^2$.

Using the factorizations of $1715, 1734$, and 2023, we can create the conditions required on x. For example, because 7^2 divides 1715, then it cannot divide $x + 17$ or else $7^2 | \gcd(x + 17, 1715)$, meaning that $2023 = 7 \cdot 17^2$ cannot divide it. In this way, our conditions are that $5 \nmid (x + 17)$, that $7^2 \nmid (x + 17)$, that $2 \nmid x$, and that $3 \nmid x$. Notice that $5 \nmid (x + 17)$ is equivalent to $x \not\equiv 3 \pmod 5$, and the other three conditions can be written similarly. So, by the Chinese Remainder Theorem, out of every $5 \cdot 7^2 \cdot 2 \cdot 3$ consecutive numbers, exactly $4 \cdot (7^2 - 1) \cdot 1 \cdot 2 = 2^7 \cdot 3$ of these numbers will satisfy all of these conditions. Therefore, out of the first $210 \cdot 2023$ positive numbers, since $210 \cdot 2023 = (5 \cdot 7^2 \cdot 2 \cdot 3) \cdot 17^2$, then $(2^7 \cdot 3) \cdot 17^2$ of these smallest positive numbers will satisfy all four of our conditions. Moreover, note that $7^2 \mid (210 \cdot 2023)$, implying that S (which excludes the number $210 \cdot 2023$) is also $2^7 \cdot 3 \cdot 17^2$. Thus, the answer is $2 \cdot 7 + 3 + 17 \cdot 2 = \boxed{51}$.

24. Define a sequence $a_0, a_1, a_2, \ldots$ recursively by $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + xa_n$ for each $n \geq 0$ and some real number x. The infinite series

$$\sum_{n=0}^{\infty} \frac{a_n}{10^n} = 1.$$

Compute x.

Answer: 80

Solution: For brevity, let S denote the summation. Notice that

$$\begin{aligned} 100S &= 100a_0 + 10a_1 + \sum_{n=0}^{\infty} \frac{a_{n+2}}{10^n} \\ &= 100a_0 + 10a_1 + \sum_{n=0}^{\infty} \frac{a_{n+1}}{10^n} + \sum_{n=0}^{\infty} \frac{xa_n}{10^n} \\ &= 100a_0 + 10a_1 + 10(S - a_0) + xS. \end{aligned}$$

Rearranging, we find

$$x = \frac{90S - 90a_0 - 10a_1}{S} = \frac{90(1) - 90(0) - 10(1)}{1} = \boxed{80}.$$

25. A tetrahedron has three edges of length 2 and three edges of length 4, and one of its faces is an equilateral triangle. Compute the radius of the sphere that is tangent to every edge of this tetrahedron.

Answer: $\mathbf{\frac{3\sqrt{11}}{11}}$

Solution: We first make the observation that the equilateral triangle face must have a side length of 2. If it were side length 4, the other edges would need to be longer than length 2 in order to converge at a fourth point. So, the tetrahedron is a pyramid with an equilateral triangle base, and we can take advantage of its symmetrical properties. By symmetry, the center of the sphere lies on the altitude to the equilateral triangle base. Let P be the apex of the pyramid, let A be a vertex of the equilateral triangle face, let M be the foot of the altitude of the equilateral triangle from point A, let G be the center of the equilateral triangle base (which is also the foot of the altitude of the pyramid from P onto the equilateral triangle base), let O be the center of the sphere, and let B be the foot of the altitude in $\triangle AOP$ from O onto $\overline{AP}$. All of these points are on the same cross-section, and we are given the lengths $PA = 4$, $AM = \sqrt{2^2 - 1^2} = \sqrt{3}$, $PM = \sqrt{4^2 - 1^2} = \sqrt{15}$. Moreover, B is the closest point to O on $\overline{AP}$ and M is the closest point to O on the edge of the equilateral triangle opposite A, so $OB = OM = r$. Since AM is a median, $AG = \frac{2}{3} \cdot \sqrt{3} = \frac{2}{\sqrt{3}}$ and $MG = \frac{1}{3} \cdot \sqrt{3} = \frac{1}{\sqrt{3}}$, and by the Pythagorean Theorem, we know that $OG = \sqrt{r^2 - \frac{1}{3}}$ and $PG = \sqrt{(\sqrt{15})^2 - \frac{1}{3}} = \sqrt{\frac{44}{3}}$, so $OP = \sqrt{\frac{44}{3}} - \sqrt{r^2 - \frac{1}{3}}$. Since $\triangle OBP \sim \triangle AGP$, we have $\frac{OP}{OB} = \frac{AP}{AG}$, or

$$\frac{\sqrt{\frac{44}{3}} - \sqrt{r^2 - \frac{1}{3}}}{r} = 2\sqrt{3}.$$

We now solve for r. Multiplying both sides by $r\sqrt{3}$ gives $\sqrt{44} - \sqrt{3r^2 - 1} = 6r$, and squaring both sides gives:

$$\begin{aligned} 44 + (3r^2 - 1) - 2\sqrt{44(3r^2 - 1)} &= 36r^2 \\ \Rightarrow 2\sqrt{132r^2 - 44} &= -33r^2 + 43. \end{aligned}$$

To simplify things, let $s = 33r^2$, so that $2\sqrt{4s-44} = -s+43$. Squaring both sides again gives:

$$16s - 176 = s^2 - 86s + 43^2$$
$$\Rightarrow s^2 - 102s + 2025 = 0.$$

Observing that $2025 = 45^2 = 3^4 \cdot 5^2$, we can quickly run through its factors to find the factorization $(s-27)(s-75) = 0$, so $s = 27$ or $s = 75$. Plugging these values of s back into $2\sqrt{4s-44} = -s+43$ reveals that $s = 75$ is extraneous, so $s = 27$. Thus, $33r^2 = 27$, and since r is positive, $r = \sqrt{\frac{27}{33}} = \boxed{\frac{3\sqrt{11}}{11}}$.