1. Find $f'\left(\frac{\pi}{4}\right)$, where $f(x) = 20\cos(x) + 23\sin(x)$.

   **Answer:** $\mathbf{\frac{3\sqrt{2}}{2}}$

   **Solution:** We can integrate $20\cos(x)$ and $23\sin(x)$ separately to get $f'(x) = -20\sin x + 23\cos x$. As $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, this becomes $(-20+23)\frac{\sqrt{2}}{2} = \boxed{\frac{3\sqrt{2}}{2}}$.

2. Compute
$$\int_0^4 (x-2)^5 + (x-2)^6 + (x-2)^7 \, \mathrm{d}x\,.$$

   **Answer:** $\mathbf{\frac{256}{7}}$

   **Solution:** We can rewrite the integral by substituting $u = x - 2$. We have that $\mathrm{d}u = \mathrm{d}x$ so changing limits of integration yields $\int_{-2}^{2} u^5 + u^6 + u^7 \, \mathrm{d}u$. Since $u^5$ and $u^7$ are odd functions, the integral of either from $-2$ to $2$ is equal to $0$, so we are left with:
$$\int_{-2}^{2} u^6 \, \mathrm{d}u = \frac{u^7}{7}\bigg|_{-2}^{2} = \frac{128}{7} - \frac{-128}{7} = \boxed{\frac{256}{7}}.$$

3. Let $A$ be the area of the region bounded by $x = 0$, $y = 0$, $x = 6$, and $y = \sqrt{kx}$, for some real number $k > 0$. If $A = 36$, compute the value of $k$.

   **Answer:** $\mathbf{\frac{27}{2}}$

   **Solution:** Note that this region defined by the four lines is simply the region under the curve $y = \sqrt{kx}$ from $x = 0$ to $x = 6$. Given that the area under this curve is 36, we have
$$36 = \int_0^6 \sqrt{kx} \, \mathrm{d}x = \left(\frac{2}{3}x\sqrt{kx}\right)_0^6 = 4\sqrt{6k}.$$

   Solving this for $k$ gives us that $k = \frac{81}{6} = \boxed{\frac{27}{2}}$.

4. An ice cube melts such that it always remains a cube, and its volume decreases at a constant rate. The initial side length of the cube is 10 inches, and it takes 50 minutes for the ice cube to completely melt. When the side length of the ice cube is 4 inches, what is the rate, in inches per minute, at which the side length of the ice cube is decreasing?

   **Answer:** $\mathbf{\frac{5}{12}}$

   **Solution:** Since the rate at which the volume decreases is constant, this rate is $\frac{10^3}{50} = 20$ cubic inches per minute.

   At a given time $t$, let $V$ be the volume of the cube, and let $s$ be the side length of the cube. Since $V = s^3$, $20 = \frac{\mathrm{d}V}{\mathrm{d}t} = 3s^2\frac{\mathrm{d}s}{\mathrm{d}t}$. Thus, $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{20}{3s^2}$, so when $s = 4$, we have $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{20}{3\cdot 4^2} = \boxed{\frac{5}{12}}$.

5. Let $f(a,b) = b^3 - a^3 + a^2b - ab^2$. There exists a real number $C$ such that regardless of the choice of nonnegative real numbers $0 = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = 4$, we have $C \leq \sum_{i=1}^{n} f(x_{i-1}, x_i)$. Compute the maximum value of $C$.

   **Answer:** $\mathbf{\frac{128}{3}}$

**Solution:** We first observe that $f(a,b) = (a^2+b^2)(b-a) = 2 \cdot \frac{(a^2+b^2)(b-a)}{2}$ so this resembles a Riemann sum. Consider the sum $\frac{1}{2}\sum_{i=1}^{n} f(x_{i-1}, x_i)$. We claim we are taking a trapezoidal integral of $y = x^2$ from $x = 0$ to $x = 4$. Indeed, we first partition $[0,4]$ into $0 < x_1 < \cdots < x_n$. Then, for each $i$, we consider the trapezoid with coordinates $(x_i, 0), (x_{i+1}, 0), (x_{i+1}, x_{i+1}^2), (x_i, x_i^2)$. We also verify that we have trapezoidal sums, which overestimate the actual area, since the function $x^2$ is convex.

Thus, the greatest lower bound for the trapezoidal sums is equal to the actual area under the curve $y = x^2$ on $x \in [0,4]$, which is

$$\int_0^4 x^2 \mathrm{d}x = \frac{x^3}{3}\bigg|_0^4 = \frac{64}{3}.$$

We double the answer to get $\boxed{\frac{128}{3}}$.

6. For a positive number $x$, let $f_0(x) = \frac{1}{x}$ and $f_n(x) = \frac{d^n}{dx^n}\left(\frac{1}{x}\right)$ for all positive integers $n$. If

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{f_n(x)},$$

compute $g(1)$.

**Answer:** $\mathbf{\frac{1}{e}}$

**Solution:** Note that $f_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$. To see this inductively, clearly this is true if $n = 0$ and, if it's true for $n$, then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n n!}{x^{n+1}}\right) = \frac{(-1)^n n! \cdot (-1) \cdot (n+1)}{x^{n+2}} = \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}$, meaning it holds for $n+1$. This means that

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(-1)^n n!} = x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = xe^{-x}.$$

Hence, $g(1) = \boxed{\frac{1}{e}}$.

7. Define a sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 24$, $a_1 = 23$, and $a_{n+2} = -a_{n+1} + 6a_n$ for $n \geq 0$. Compute

$$\sum_{n=1}^{\infty} \frac{a_n}{n6^n}.$$

**Answer:** $\mathbf{14 \ln\left(\frac{3}{2}\right)}$

**Solution:** We first seek a formula for $a_i$. From the recurrence relation, we have that $x^2 = -x+6$ has 2 and $-3$ as roots, so the recurrence relation will be of the form $a_n = x2^n + y(-3)^n$ for some $x$ and $y$. We see that $a_0 = x + y = 24$, and $a_1 = 2x - 3y = 23$, so $5x = 3(x+y) + (2x-3y) = 95$, so $x = 19$ and $y = 5$.

Next, we split into two sums:

$$\sum_{n=1}^{\infty} \frac{x}{n3^n} + \sum_{n=1}^{\infty} \frac{y}{n(-2)^n}.$$

Using the Taylor expansion for $\ln(1-x)$ we get the two sums equal to $-x \ln \frac{2}{3} - y \ln \frac{3}{2}$. This is equal to $(x-y)\ln \frac{3}{2}$. Plugging in $x = 19$ and $y = 5$, and we have $S = \boxed{14 \ln \frac{3}{2}}$.

8. Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function such that

$$I(f) = \int_1^\infty \left(\sqrt{2023}xe^{-x}f(x) - \frac{1}{4}x^2 f(x)^2\right) \mathrm{d}x$$

converges and is maximized over all continuous functions on $[1, \infty) \to \mathbb{R}$. Compute $f(1) + I(f)$.

**Answer:** $\boldsymbol{\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}}$

**Solution:** Consider the function $-(axf(x) - be^{-x})^2$ for some constants $a$ and $b$. Expanding, we get the following expression:

$$2abxe^{-x}f(x) - a^2x^2f(x)^2 - b^2e^{-2x}$$

So, we get the following system of equations:

$$2ab = \sqrt{2023}$$
$$a^2 = \frac{1}{4}$$

which gives the two solutions $(a, b) = \left(\frac{1}{2}, \sqrt{2023}\right), \left(-\frac{1}{2}, -\sqrt{2023}\right)$. It is apparent that both ordered pairs yield the same result. Substituting $(a, b) = \left(\frac{1}{2}, \sqrt{2023}\right)$ yields:

$$I(f) = \int_1^\infty \left(2023e^{-2x} - \left(\frac{1}{2}xf(x) - \sqrt{2023}e^{-x}\right)^2\right) \mathrm{d}x\,.$$

Because $-\left(\frac{1}{2}xf(x) - \sqrt{2023}e^{-x}\right)^2 \leq 0$ for all $x \geq 1$, $I(f)$ is be maximized when $f(x) = \frac{2\sqrt{2023}}{xe^x}$. Setting $f$ equal to that, we get $f(1) = \frac{2\sqrt{2023}}{e}$ and

$$I(f) = \int_1^\infty 2023e^{-2x}\mathrm{d}x = \frac{2023}{2e^2},$$

leaving a final answer of $f(1) + I(f) = \boxed{\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}}$.

9. Compute

$$\int_0^{2\pi} (\sin(x) + \cos(x))^6 \,\mathrm{d}x\,.$$

**Answer:** $\boldsymbol{5\pi}$

**Solution:** We will use periodic cancelling technique. Since the bounds are from 0 to $2\pi$, any terms in the integrand like $\sin(nx)$ or $\cos(nx)$ will contribute nothing to the final answer for any natural number $n$. After expanding out $(\sin(x) + \cos(x))^6$, the only terms that survive are

$$\int_0^{2\pi} \left[\sin^6(x) + 15\sin^4(x)\cos^2(x) + 15\sin^2(x)\cos^4(x) + \cos^6(x)\right] \mathrm{d}x\,.$$

This is because squared sines and squared cosines leave off constants due to the identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$. Now, for sake of simplicity, we will now sometimes abbreviate

$\sin(x)$ with $s$ and $\cos(x)$ with $c$. Notice that $s^6+c^6=(s^2+c^2)(s^4-s^2c^s+c^4)=(s^4-s^2c^2+c^4)$. Also, $s^4c^2+s^2c^4=s^2c^2(s^2+c^2)=s^2c^2$. So now we have

$$\int_0^{2\pi}\left[\sin^4(x)+\cos^4(x)+14\sin^2(x)\cos^2(x)\right]\mathrm{d}x$$

With more trig manipulation, we have $s^4+c^4+2s^2c^2=(s^2+c^2)^2=1 \implies s^4+c^4=1-2s^2c^2$. This simplifies our integral to

$$\begin{aligned}\int_0^{2\pi}\left[1+12\sin^2(x)\cos^2(x)\right]dx &= \int_0^{2\pi}\left[1+3\sin^2(2x)\right]\mathrm{d}x\\ &= \int_0^{2\pi}\left[1+\frac{3}{2}\left(1-\cos(4x)\right)\right]\mathrm{d}x = 2\pi\left(1+\frac{3}{2}\right)=\boxed{5\pi}.\end{aligned}$$

10. Compute

$$\int_0^{\infty}\frac{\sin(x)}{x^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}\,\mathrm{d}x\,.$$

**Answer:** $\boldsymbol{\frac{\pi}{2}(e-1)}$

**Solution:** Using the trigonometric product-to-sum formulas and then integration by parts, we have that:

$$\begin{aligned}\int_0^{\infty}\frac{\sin(x)}{x^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}\,\mathrm{d}x &= \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\sin(x)\sin(nx)}{x^2}\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos((n-1)x)-\cos((n+1)x)}{x^2}\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\left(\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x}\right)\right]\left(\cos((n-1)x)-\cos((n+1)x)\right)\mathrm{d}x\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{(n+1)\sin((n+1)x)-(n-1)\sin((n-1)x)}{x}\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\left(\frac{n+1}{n!}\int_0^{\infty}\frac{\sin((n+1)x)}{x}\,\mathrm{d}x\right)-\left(\frac{n-1}{n!}\int_0^{\infty}\frac{\sin((n-1)x)}{x}\,\mathrm{d}x\right)\right)\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\pi}{2}(n+1)-\frac{\pi}{2}(n-1)\right]\\ &= \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\\ &= \boxed{\frac{\pi}{2}(e-1)}\end{aligned}$$

as desired.