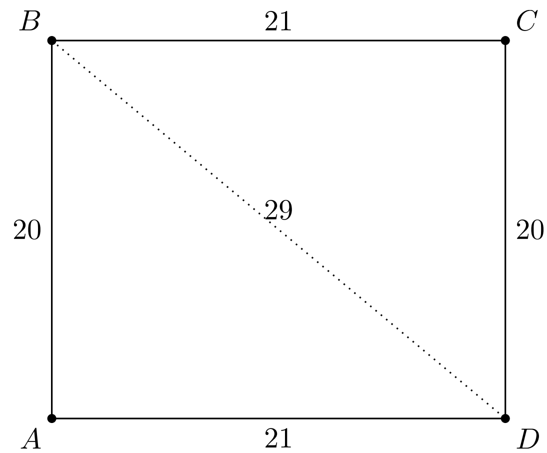
1. Shreyas has a rectangular piece of paper $ABCD$ such that $AB = 20$ and $AD = 21$. Given that Shreyas can make exactly one straight-line cut to split the paper into two pieces, compute the maximum total perimeter of the two pieces.

   Answer: 140

   Solution:

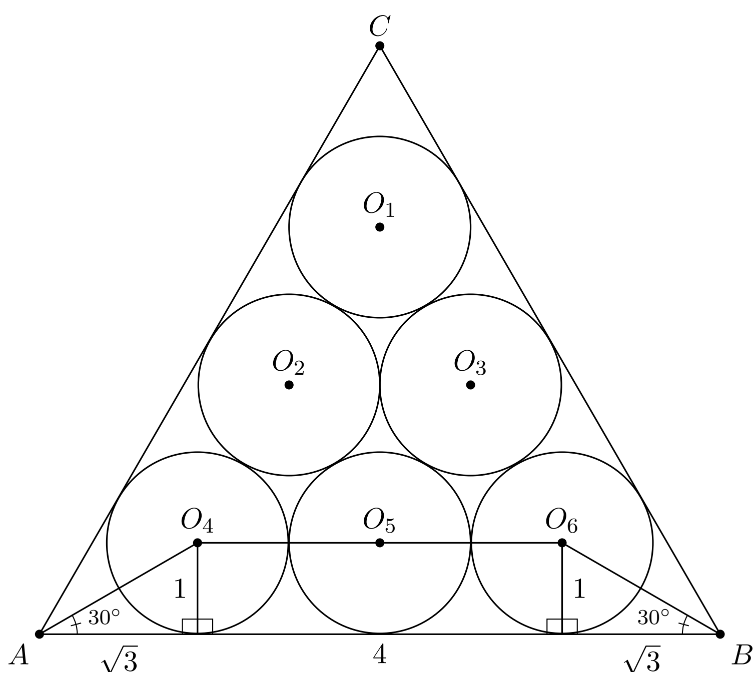


The total perimeter increases by twice the length of the cut made. The maximum length of a cut is a diagonal of the rectangle $BD = \sqrt{20^2 + 21^2} = 29$, so our answer is $2\cdot20+2\cdot21+2\cdot29 = \boxed{140}$.

2. Compute the area of the smallest triangle which can contain six congruent, non-overlapping unit circles.

   Answer: $\mathbf{12 + 7\sqrt{3}}$

   Solution:



The optimal strategy is to stack the circles in a pyramid as shown in the diagram above. We can construct a rectangle by connecting the centers of circles O_4 and O_6 with a segment having its midpoint at the center of O_5. Then, we can complete the rectangle by connecting O_4 and O_6 to the base $\overline{AB}$. Note that because each circle has radius 1, the rectangle has a height of 1 and a length of 4. This process also creates two $30-60-90$ triangles with height 1. Therefore, we can calculate the length of $\overline{AB}$ to be $4+2\sqrt{3}$. So, the area of $\triangle ABC$ is $\frac{(4+2\sqrt{3})^2\sqrt{3}}{4} = \boxed{12+7\sqrt{3}}$.

3. In quadrilateral $ABCD$, suppose that $\overline{CD}$ is perpendicular to $\overline{BC}$ and $\overline{DA}$. Point E is chosen on segment $\overline{CD}$ such that $\angle AED = \angle BEC$. If $AB = 6$, $AD = 7$, and $\angle ABC = 120°$, compute $AE + EB$.

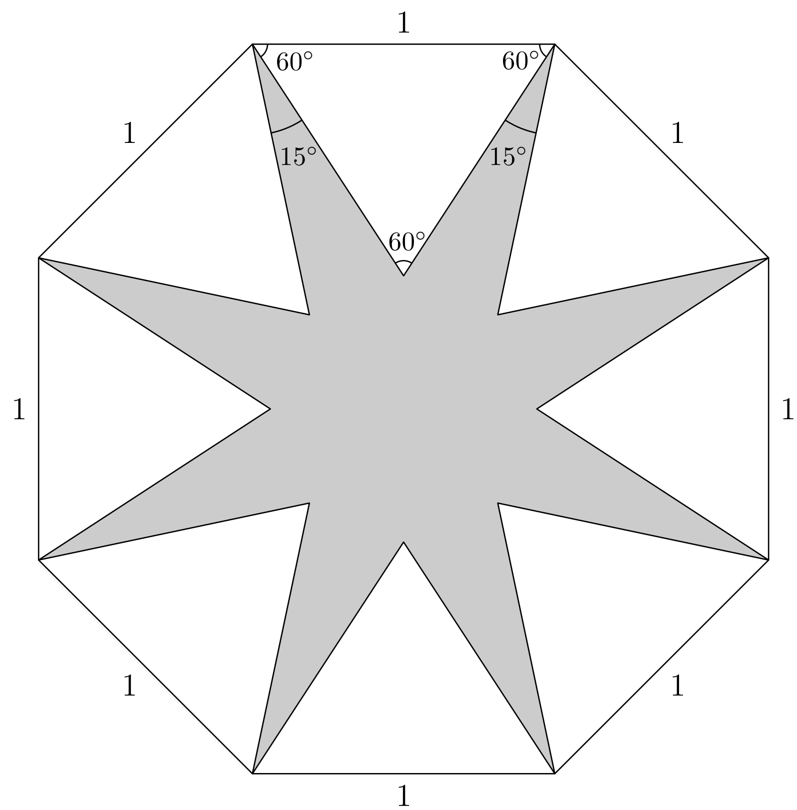
   Answer: $\mathbf{2\sqrt{37}}$

   Solution: Since $\angle ABC = 120°$, $\angle BAD = 60°$, so $BC = 4$. Let A' and B' be the reflection of A and B over the segment $\overline{CD}$, respectively. Then note that $AE + EB = AB'$. Letting F be the projection of B' onto AA', we have that $FB' = \sqrt{6^2 - 3^2} = 3\sqrt{3}$, and $FA = A'A - A'F = 11$, so $AE + EB = \sqrt{11^2 + (3\sqrt{3})^2} = \boxed{2\sqrt{37}}$.

4. An equilateral polygon has unit side length and alternating interior angle measures of 15° and 300°. Compute the area of this polygon.

   Answer: $\mathbf{2+2\sqrt{2}-2\sqrt{3}}$

   Solution: The average interior angle measure is $\frac{15°+300°}{2} = \frac{315}{2}°$, so the average exterior angle measure is $\frac{45}{2}°$, indicating that this polygon has 16 sides. Guided by this, we can then draw the polygon, as follows.



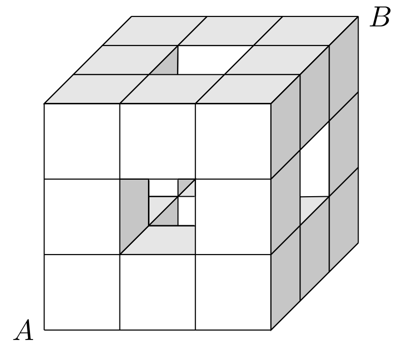
The given angles tell us that the polygon should look like an 8-pointed star. Upon drawing the shape, we find that the outer 8 points form a regular octagon, and when we draw the octagon, we find that the 8 triangles we subtract off to form the 8 pointed star are all equilateral triangles! So, the octagon also has unit side length. The area of the octagon can be taken by extending side lengths to form a square, then subtract off the four isosceles right triangles; this gives an area of $(\sqrt{2}+1)^2-4\cdot\frac{1}{4}=2\sqrt{2}+2$. The area of the 8 pointed star is $2\sqrt{2}+2-8\cdot\frac{\sqrt{3}}{4}=\boxed{2+2\sqrt{2}-2\sqrt{3}}$.

5. Let circles ω_1 and ω_2 intersect at P and Q. Let the line externally tangent to both circles that is closer to Q touch ω_1 at A and ω_2 at B. Let point T lie on segment $\overline{PQ}$ such that $\angle ATB = 90^\circ$. Given that $AT = 6$, $BT = 8$, and $PT = 4$, compute PQ.

Answer: $\mathbf{\frac{56}{9}}$

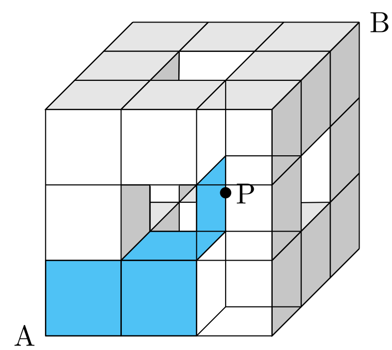
Solution: Let $\overline{PQ}$ intersect $\overline{AB}$ at M. By Power of a Point, $MA^2 = MQ\cdot MP = MB^2$, thus $MA = MB$. By the Pythagorean Theorem, we get $AB = 10$. So, $AM = BM = 5$. Also, since M is the midpoint of the hypotenuse of $\triangle ATB$, $MT = 5$. So, we have that $MP = MT+TP = 9$. Plugging this into the Power of a Point equation we had earlier gives $5^2 = MQ\cdot 9 \implies MQ = \frac{25}{9}$. So, $PQ = MP - MQ = 9 - \frac{25}{9} = \boxed{\frac{56}{9}}$.

6. Consider 27 unit-cubes assembled into one $3\times3\times3$ cube. Let A and B be two opposite corners of this large cube. Remove the one unit-cube not visible from the exterior, along with all six unit-cubes in the center of each face. Compute the minimum distance an ant has to walk along the surface of the modified cube to get from A to B.

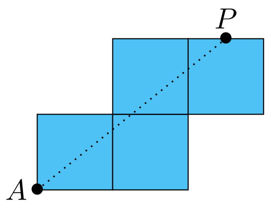


Answer: $\mathbf{\sqrt{41}}$

Solution:



On a normal cube, the ant would like to walk on a straight line on the net of the cube. In this special cube with holes in the middle, the ant would ideally want to travel as spacially close to the space diagonal as possible while still traveling in a straight line on the net. To do this, it can travel in two separate straight line segments from A to P to B, where P is the midpoint of the edge it lies on. By symmetry, notice how the path from A to P is the same length as the path from P to B. So, our answer is just twice the length of the path from A to P. We can unfold the net as shown below.

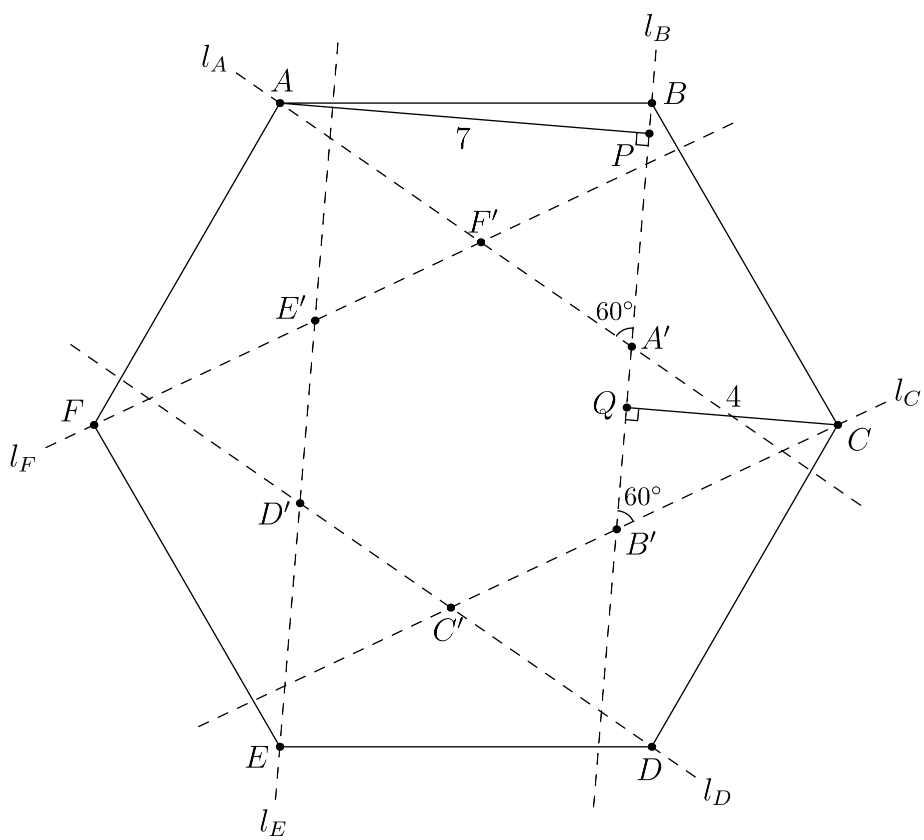


Here, we can see that the length of the path from A to P is $\sqrt{2^2 + \frac{5}{2}^2} = \frac{\sqrt{41}}{2}$. So, our answer is $2 \cdot \frac{\sqrt{41}}{2} = \boxed{\sqrt{41}}$.

7. The line l passes through vertex B and the interior of regular hexagon $ABCDEF$. If the distances from l to the vertices A and C are 7 and 4, respectively, compute the area of hexagon $ABCDEF$.

Answer: $\mathbf{74\sqrt{3}}$

Solution: We have the following diagram.



Consider the 6 lines created when l is rotated about the center of the hexagon in 60° increments, and label them $l_A, l_B, \dots,$ and l_F, where l_P passes through vertex P (note that $l = l_B$). Let A' be the intersection of l_A and l_B, let B' be the intersection of l_B and l_C, and define similarly C', D', E', F'. Then, by symmetry, $\triangle AA'B \cong \triangle BB'C \cong \dots \cong \triangle FF'A$ and $A'B'C'D'E'F'$ is a regular hexagon, so $\angle AA'B = \angle BB'C = \dots = \angle FF'A = 60^\circ$. Then, the hexagon is partitioned into six congruent triangles and a smaller regular hexagon, so we will find their side lengths to get their areas.

Let P be the foot of the altitude from A onto l_B and Q be the foot of the altitude from C onto l_B, so that $AP = 7$ and $CQ = 4$. Then $\triangle APA'$ and $\triangle CQB'$ are 30-60-90 triangles, so $AA' = \frac{2}{\sqrt{3}} \cdot AP = \frac{14}{\sqrt{3}}$ and $CB' = \frac{2}{\sqrt{3}} \cdot CQ = \frac{8}{\sqrt{3}}$. Since $\triangle AA'B \cong \triangle BB'C$, $BA' = CB' = \frac{8}{\sqrt{3}}$, and since $\triangle BB'C \cong \triangle FF'A$, $F'A' = AA' - AF' = AA' - CB' = \frac{14}{\sqrt{3}} - \frac{8}{\sqrt{3}} = 2\sqrt{3}$. Then by the sine area formula, the area of $\triangle AA'B$ is $\frac{1}{2}(AA')(BA')\sin 60^\circ = \frac{28}{\sqrt{3}}$. We also know that the area of $A'B'C'D'E'F'$ is $6 \cdot \frac{(FA')^2\sqrt{3}}{4} = 18\sqrt{3}$, so the area of $ABCDEF$ is $6[\triangle AA'B] + [A'B'C'D'E'F'] = 56\sqrt{3} + 18\sqrt{3} = \boxed{74\sqrt{3}}$.

8. Let $\triangle ABC$ be a triangle with $AB = 15$, $AC = 13$, $BC = 14$, and circumcenter O. Let l be the line through A perpendicular to segment $\overline{BC}$. Let the circumcircle of $\triangle AOB$ and the circumcircle of $\triangle AOC$ intersect l at points X and Y (other than A), respectively. Compute the length of $\overline{XY}$.

   Answer: $\mathbf{\frac{112}{15}}$

   Solution: Let D be the foot of the altitude from A to $\overline{BC}$. Let H' be the intersection of $\overline{AD}$ with the circumcircle of triangle $\triangle ABC$. Using directed angles, we know $\measuredangle XYO = \measuredangle AYO = \measuredangle ACO = \measuredangle OAC = \measuredangle BAH' = \measuredangle BCH'$. Similarly, $\measuredangle YXO = \measuredangle AXO = \measuredangle ABO = \measuredangle OAB = \measuredangle CAH' = \angle CBH'$. So, we know that $\triangle H'BC$ is similar to $\triangle OXY$. We can calculate the similarity ratio by calculating the ratio of the lengths of the altitudes. Let M be the midpoint of $\overline{BC}$. We can calculate $DC = 5$, $BD = 9$, and $AD = 12$. We know the altitude from O to $\overline{XY}$ has length $MC - DC = 7 - 5 = 2$. We can calculate the altitude from H' to $\overline{BC}$ using Power of a Point to be $\frac{5 \cdot 9}{12} = \frac{15}{4}$. So, using the similarity ratio, $XY = \frac{2}{\frac{15}{4}} \cdot 14 = \boxed{\frac{112}{15}}$.

9. Let $ABCD$ be a convex quadrilateral such that $\triangle ABC$ is equilateral. Let P be a point inside the quadrilateral such that $\triangle APD$ is equilateral and $\angle PCD = 30^\circ$. Given that $CP = 2$ and $CD = 3$, compute the area of the triangle formed by P, the midpoint of segment $\overline{BC}$, and the midpoint of segment $\overline{AB}$.

   Answer: $\mathbf{\frac{5\sqrt{3}+12}{16}}$

   Solution: Since $AB = AC$, $AP = AD$, and $\angle BAP = \angle CAD$, $\triangle APB$ is congruent to $\triangle ADC$, so $BP = 3$. Also $\angle BPC = 360^\circ - \angle APB - \angle APC = 360^\circ - \angle APC - \angle ADC = \angle DAP + \angle DCP = 90^\circ$, so $\angle BPC$ is right. The side length of $\triangle ABC$ is then $\sqrt{2^2 + 3^2} = \sqrt{13}$. Let M be the midpoint of $\overline{BC}$ and N be the midpoint of $\overline{AB}$. We then have $MN = MP = \frac{\sqrt{13}}{2}$. Now, it simply suffices to find $\sin(\angle NMP)$. This is the same as

$$\begin{aligned}\sin(180^\circ - \angle BMN - \angle PMC) &= \sin(120^\circ - \angle PMC) \\ &= \sin(120^\circ)\cos(\angle PMC) - \cos(120^\circ)\sin(\angle PMC).\end{aligned}$$

   We can calculate $\sin(\angle PMC) = \sin(2\angle PBC) = 2\sin(\angle PBC)\cos(\angle PBC) = 2 \cdot \frac{2}{\sqrt{13}} \cdot \frac{3}{\sqrt{13}} = \frac{12}{13}$.

From this we get $\cos(\angle PMC) = \frac{5}{13}$. Thus, our answer is

$$\frac{1}{2} \cdot \left(\frac{\sqrt{13}}{2}\right)^2 \cdot \left(\frac{\sqrt{3}}{2} \cdot \frac{5}{13} + \frac{1}{2} \cdot \frac{12}{13}\right) = \frac{13}{8} \cdot \frac{5\sqrt{3}+12}{26} = \boxed{\frac{5\sqrt{3}+12}{16}}.$$

10. Consider $\triangle ABC$ such that $CA + AB = 3BC$. Let the incircle ω touch segments $\overline{CA}$ and $\overline{AB}$ at E and F, respectively, and define P and Q such that segments $\overline{PE}$ and $\overline{QF}$ are diameters of ω. Define the function $\mathcal{D}$ of a point K to be the sum of the distances from K to P and Q (i.e. $\mathcal{D}(K) = KP + KQ$). Let W, X, Y, and Z be points chosen on lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CE}$, $\overleftrightarrow{EF}$, and $\overleftrightarrow{FB}$, respectively. Given that $BC = \sqrt{133}$ and the inradius of $\triangle ABC$ is $\sqrt{14}$, compute the minimum value of $\mathcal{D}(W) + \mathcal{D}(X) + \mathcal{D}(Y) + \mathcal{D}(Z)$.

Answer: $\mathbf{24\sqrt{2}}$

Solution: Define I to be the incenter, r to be the inradius, I_a, I_b, I_c to be the A, B, C-excenters respectively, and r_a to be the A-exradius. Also, let M be the midpoint of $\overline{EF}$. Then, we have $\frac{r_a}{r} = \frac{b+c+a}{b+c-a} = 2$, meaning that the exradius is twice the inradius. By homothety, this means that I is the midpoint of $\overline{AI_a}$, and since I is the orthocenter of $\triangle I_aI_bI_c$, I is also the midpoint of the altitude $\overline{AI_a}$. Let D be the point of tangency of ω to $\overline{BC}$. Then, note that since I is the circumcenter of $\triangle DEF$, I is also the orthocenter of the medial triangle of $\triangle DEF$. Since $\triangle DEF$ is homothetic to $\triangle I_aI_bI_c$, the medial triangle is homothetic to $\triangle I_aI_bI_c$ about I, meaning that I must lie on the midpoint of the altitude from M to the D-midline of $\triangle DEF$. However, since $\overline{PQ}$ is the reflection of $\overline{EF}$ over I, $\overleftrightarrow{PQ}$ must contain the D-midline. Then, note that the midpoints B' and C' of $\overline{DF}$ and $\overline{DE}$, respectively, must lie on this midline. Taking inversion $\mathcal{I}$ about ω, since $\mathcal{I}(B') = B, \mathcal{I}(C') = C$, as well as B, C, I, and I_a concyclic, we can conclude that P, Q, B, C, I, and I_a are concyclic.

Now, let D' be the intersection of $\overleftrightarrow{AI}$ and $\overleftrightarrow{BC}$, and let N be the second intersection of the circumcircle of $\triangle PQD'$ and $\overleftrightarrow{BC}$. By directed angles, we have

$$\measuredangle BNP = \measuredangle D'NP = \measuredangle D'QP = \measuredangle QPD' = \measuredangle QND' = \measuredangle QNC$$

Thus, there exists an ellipse with foci P and Q tangent to $\overline{BC}$ at N, since $\mathcal{D}(W)$ is minimal when $W = N$. Now, we prove the following lemma:

Lemma. Given a triangle $\triangle ABC$, there exists a unique ellipse with foci P, and Q tangent to $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$ if P and Q are isogonal conjugates in $\triangle ABC$.

Proof. Given that P, Q are isogonal conjugates in $\triangle ABC$, let D, E, and F be the feet from P to $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$ respectively, and define X, Y, and Z similarly for Q. Additionally, let M be the midpoint of PQ. For a point K on line l that minimizes $PK + KQ$, we have that $PK + KQ = \sqrt{(PP' + QQ')^2 + P'Q'^2} = 2MP' = 2MQ'$ where P' and Q' are the feet of P and Q on line l.

It suffices to show that M is the center of both $\odot(DEF)$ and $\odot(XYZ)$. We have $MD = MX$, $ME = MY$, and $MF = MZ$ since M lies on the perpendicular bisectors of $\overline{DX}$, $\overline{EY}$, and $\overline{FZ}$. We also note that $\overleftrightarrow{AP} \perp \overleftrightarrow{YZ}$. This is true since

$$\measuredangle(AP, YZ) = \measuredangle PAZ + \measuredangle AZY = \measuredangle YAQ + \measuredangle AQY = \measuredangle QYA = \frac{\pi}{2}$$

and $AYQZ$ is cyclic. Let K be the foot from A to $\overleftrightarrow{YZ}$, so since $\measuredangle PKZ = \measuredangle PFZ = \frac{\pi}{2}$ and $\measuredangle PKY = \measuredangle PEY = \frac{\pi}{2}$, $PKEY$ and $PKFZ$ are cyclic.

Then note $AF \cdot AZ = AP \cdot AK = AE \cdot AY$ meaning that $EFYZ$ is cyclic. Since M lies on the perpendicular bisectors of $\overline{EY}$ and $\overline{FZ}$, M is the circumcenter, and by similar reasoning, M is equidistant from D, E, F, X, Y, and Z. Let this common distance be R. Define T, U, and V as the points on $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$ such that $PT + TQ, PU + UQ, PV + VQ$ are all minimal. We have $PT + TQ = 2ME = 2R$, and similarly, $PU + UQ = PV + VQ = 2R$. Thus, the ellipse $\mathcal{E}$, foci P, Q tangent to $\overleftrightarrow{BC}$ must be tangent at T since T minimizes the sum $PT + TQ$. However, since $PT + TQ = PU + UQ = PV + VQ = 2R$, the ellipse passes through U and V. Since the ellipse foci P, Q passing through U, and the ellipse foci P, Q tangent to $\overleftrightarrow{CA}$ are both unique, they must be the same ellipse and thus, $\mathcal{E}$ is tangent to $\overleftrightarrow{CA}$ as well. And by similar reasoning, $\mathcal{E}$ is tangent to $\overleftrightarrow{AB}$ at V.

Finally, to prove uniqueness we simply note that the point K on line l minimizing $PK + KQ$ is unique, and therefore there can only be one possible point of tangency of the ellipse to l. Since an ellipse can be uniquely determined by its foci and major axis, $\mathcal{E}$ must be unique, completing the proof. $\square$

Now, note that $\odot(BCPQII_a)$ has a center on $\overleftrightarrow{AI}$. Then, let B_1, C_1 be the second intersections of $\odot(BCPQII_a)$ with $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$ respectively. Thus, B_1, C_1 are the reflections of B, C over $\overleftrightarrow{AI}$ respectively. Now, we have

$$\measuredangle ABP = \measuredangle AB_1Q = \measuredangle CB_1Q = \measuredangle CBQ$$

$$\measuredangle ACQ = \measuredangle AC_1P = \measuredangle BC_1P = \measuredangle BCP$$

so P, Q are isogonal conjugates in $\triangle ABC$. We also have

$$\measuredangle AFQ = \measuredangle AFI = \frac{\pi}{2} = \measuredangle EFP$$

$$\measuredangle AEP = \measuredangle AEI = \frac{\pi}{2} = \measuredangle FEQ$$

so P and Q are also isogonal conjugates in $\triangle AEF$. Denote $\mathcal{E}$ to be the ellipse of foci P and Q tangent to $\overline{BC}$ at N. By our lemma, $\mathcal{E}$ is also tangent to $\overline{CA}$, $\overline{AB}$, and $\overline{EF}$. Since P and Q are symmetric about $\overleftrightarrow{AI}$, $\mathcal{E}$ is as well, meaning the tangency point on $\overline{EF}$ is M. Finally, note that $AE = AF = \frac{b+c-a}{2} = a = BC$. We have

$$\min(\mathcal{D}(W) + \mathcal{D}(X) + \mathcal{D}(Y) + \mathcal{D}(Z)) = 4\mathcal{D}(M) = 8QM = 8\sqrt{EM^2 + (2IM)^2}$$

$$8\sqrt{EI^2 + 3IM^2} = 8\sqrt{r^2 + 3\left(\frac{r^2}{\sqrt{r^2 + a^2}}\right)^2} = 8r\sqrt{\frac{4 + \frac{a^2}{r^2}}{1 + \frac{a^2}{r^2}}} = 8\sqrt{14 \cdot \frac{4 + \frac{19}{2}}{1 + \frac{19}{2}}} = \boxed{24\sqrt{2}}$$