1. Carson and Emily attend different schools. Emily's school has four times as many students as Carson's school. The total number of students in both schools combined is 10105. How many students go to Carson's school?

   **Answer: 2021**

   **Solution:** Let $C$ be the number of students that are in Carson's school, $E$ the number of students in Emily's. Note that $E = 4C$, therefore the given total of students is $C + E = C + 4C = 5C = 10105$. Divison reveals that $C = \boxed{2021}$.

2. Let $x$ be a real number such that $x^2 - x + 1 = 7$ and $x^2 + x + 1 = 13$. Compute the value of $x^4$.

   **Answer: 81**

   **Solution:** Subtracting the first equation from the second yields $2x = 6$, which implies $x = 3$. Thus, $x^4 = 3^4 = \boxed{81}$.

3. A scalene acute triangle has angles whose measures (in degrees) are whole numbers. What is the smallest possible measure of one of the angles, in degrees?

   **Answer: 3**

   **Solution:** To minimize one of the angles, we need to maximize the other two. Since the triangle is acute, all angles are at most 89°. But since the triangle is scalene, we can't have two 89° angles, so we ought to have an 89° angle and an 88° angle. The measure of the remaining angle is then $180° - 89° - 88° = \boxed{3°}$.

4. Moor and Samantha are drinking tea at a constant rate. If Moor starts drinking tea at 8:00am, he will finish drinking 7 cups of tea by 12:00pm. If Samantha joins Moor at 10:00am, they will finish drinking the 7 cups of tea by 11:15am. How many hours would it take Samantha to drink 1 cup of tea?

   **Answer:** $\mathbf{\frac{20}{21}}$

   **Solution:** Let $m$ and $s$ be the number of cups of tea per hour that Moor and Samantha can drink, respectively. Then $m = \frac{7}{4}$ since Moor can drink 7 cups in 4 hours. To figure out $s$, note that from 10am to 11:15 am, Samantha will have spent 1.25 hours drinking tea while Moor will have spent 3.25 hours drinking tea in total. Solving for $s$ using

   $$3.25m + 1.25s = 7,$$

   we get that $s = 1.05 = \frac{21}{20}$. Thus, it would take Samantha $\frac{1}{s} = \boxed{\frac{20}{21}}$ hours to drink 1 cup of tea.

5. Bill divides a $28 \times 30$ rectangular board into two smaller rectangular boards with a single straight cut, so that the side lengths of both boards are positive whole numbers. How many different pairs of rectangular boards, up to congruence and arrangement, can Bill possibly obtain? (For instance, a cut that is 1 unit away from either of the edges with length 28 will result in the same pair of boards: either way, one would end up with a $1 \times 28$ board and a $29 \times 28$ board.)

   **Answer: 29**

   **Solution:** Either Bill cuts the board parallel to the side with length 28, or cuts the board parallel to the side with length 30. In the first case, one of the side lengths for both boards is 28, and the other side length of the *smaller* board can be a whole number from 1 to 15. In the second case, one of the side lengths for both boards is 30, and the other side length of the smaller board can be a whole number can be a whole number from 1 to 14. Thus the number of possible combinations is $15 + 14 = \boxed{29}$.

6. A toilet paper roll is a cylinder of radius 8 and height 6 with a hole in the shape of a cylinder of radius 2 and the same height. That is, the bases of the roll are annuli with inner radius 2 and outer radius 8. Compute the surface area of the roll.

   **Answer: $240\pi$**

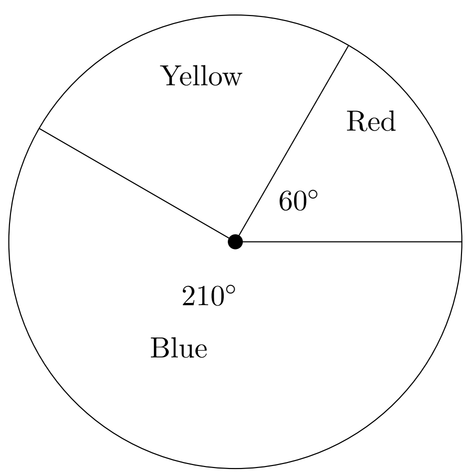
   **Solution:** The surface area of the toilet paper roll has three components: the circular bases, the external lateral area, and the internal lateral area. The area of the circular bases are $2\pi(8^2 - 2^2) = 120\pi$. The external lateral area is $2 \cdot \pi \cdot 8 \cdot 6 = 96\pi$, and the internal lateral area is $2 \cdot \pi \cdot 2 \cdot 6 = 24\pi$. Summing these results gives a total surface area of $\boxed{240\pi}$.

7. Alice is counting up by fives, starting with the number 3. Meanwhile, Bob is counting down by fours, starting with the number 2021. How many numbers between 3 and 2021, inclusive, are counted by both Alice and Bob?

   **Answer: 101**

   **Solution:** We see that 2021 divided by 4 has remainder 1, thus Bob counts all numbers that leave remainder 1 when divided by 4 (1 mod 4). Alice counts all numbers that leave remainder 3 when divided by 5 (3 mod 5). To be counted by both Alice and Bob, the number must leave remainder 13 when divided by 20 (13 mod 20). There are $\boxed{101}$ such numbers, starting from $20 \cdot 0 + 13$ and ending at $20 \cdot 100 + 13$.

8. On the first day of school, Ashley the teacher asked some of her students what their favorite color was and used those results to construct the pie chart pictured below. During this first day, 165 students chose yellow as their favorite color. The next day, she polled 30 additional students and was shocked when none of them chose yellow. After making a new pie chart based on the combined results of both days, Ashley noticed that the angle measure of the sector representing the students whose favorite color was yellow had decreased. Compute the difference, in degrees, between the old and the new angle measures.




   **Answer: $\frac{90}{23}^\circ$**

   **Solution:** First we figure out the total number of kids already surveyed on the first day. Since the angles of the sectors add up to 360°, we find that the angle of the sector in the first day pie chart is $360 - (210 + 60) = 90°$. That means that $\frac{1}{4}$ of the kids already surveyed chose yellow as favorite color, so Ashley already surveyed $165 \cdot 4 = 660$ kids on the first day.

Then we figure out the angle measure of the sector representing yellow in Ashley's new pie chart. Now there are $660 + 30 = 690$ kids who are surveyed, and 165 of them chose yellow as favorite color (since none of the newer kids chose yellow). Thus, the angle representing the sector is $\frac{165}{690} \cdot 360 = \frac{1980}{23}^\circ$, and so the difference of the angle measure is $90 - \frac{1980}{23} = \boxed{\frac{90}{23}^\circ}$.

9. Rakesh is flipping a fair coin repeatedly. If $T$ denotes the event where the coin lands on tails and $H$ denotes the event where the coin lands on heads, what is the probability Rakesh flips the sequence $HHH$ before the sequence $THH$?

**Answer:** $\frac{1}{8}$

**Solution:** The entire problem relies on the first three flips. If Rakesh doesn't flip $HHH$ in the first three flips, then it will be impossible for him to flip $HHH$ before $THH$ at any point. One way to see this is that getting $HHH$ requires getting $HH$ first but at any point if we have a $T$ then $THH$ will happen before $HHH$. The probability Rakesh flips $HHH$ in the first three flips is $\boxed{\frac{1}{8}}$.

10. Triangle $\triangle ABC$ has side lengths $AB = AC = 27$ and $BC = 18$. Point $D$ is on $\overline{AB}$ and point $E$ is on $\overline{AC}$ such that $\angle BCD = \angle CBE = \angle BAC$. Compute $DE$.

**Answer:** 10

**Solution:** Since $\triangle ABC$ is isosceles, $\angle ABC = \angle ACB$. By AA similarity, $\triangle BCD \sim \triangle BAC \sim \triangle CBE$, and so by ASA congruence, $\triangle BCD \cong \triangle CBE$. From similarity, we get $\frac{BD}{BC} = \frac{BC}{BA} = \frac{2}{3}$, so $BD = 18 \cdot \frac{2}{3} = 12$, and therefore $CE = BD = 12$. Subtracting, we can note that $AD = AB - BD = 15$ and $AE = AC - CE = 15$. Since $\frac{AD}{AB} = \frac{AE}{AC}$, we can conclude that $DE$ and $BC$ are parallel. Finally by similarity, we obtain that $\frac{DE}{BC} = \frac{AD}{AB}$ meaning $DE = BC \cdot \frac{AD}{AB} = 18 \cdot \frac{5}{9} = \boxed{10}$.

11. Compute the number of sequences of five positive integers $a_1, \ldots, a_5$ where all $a_i \leq 5$ and the greatest common divisor of all five integers is 1.

**Answer:** 3091

**Solution:** We use complementary counting. There are $5^5$ sequences of integers in total, $2^5$ sequences with GCD divisible by 2, $1^5$ sequence with GCD divisible by 3, and $1^5$ sequence with GCD divisible by 5. Therefore, there are $5^5 - 2^5 - 1^5 - 1^5 = \boxed{3091}$ such sequences.

12. Let $a$, $b$, and $c$ be the solutions of the equation

$$x^3 - 3 \cdot 2021^2 x = 2 \cdot 2021^3.$$

Compute $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

**Answer:** $-\frac{3}{4042}$

**Solution:** Noting that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ca+ab}{abc}$, we can use Vieta's Theorem to get $bc + ca + ab = -3 \cdot 2021^2$ and $abc = 2 \cdot 2021^3$. By dividing, we get $-\frac{3 \cdot 2021^2}{2 \cdot 2021^3} = \boxed{-\frac{3}{4042}}$.

13. A six-sided die is rolled four times. What is the probability that the minimum value of the four rolls is 4?

**Answer:** $\frac{65}{1296}$

**Solution:** The probability that the minimum roll is greater than 4 is $\left(\frac{2}{6}\right)^4$. The probability that the minimum roll is greater than 3 is $\left(\frac{3}{6}\right)^4$. Since the minimum roll being equal to 4 is the same as the minimum being greater than 3 but not greater than 4, the desired probability is $\frac{3^4-2^4}{6^4} = \boxed{\frac{65}{1296}}$.

14. Let $r_1, r_2, \ldots, r_{47}$ be the roots of $x^{47} - 1 = 0$. Compute

$$\sum_{i=1}^{47} r_i^{2020}.$$

**Answer:** 0

**Solution:** Observe that

$$r_i^{47} = 1 \implies r_i^{2021} = 1 \implies r_i^{2020} = \frac{1}{r_i},$$

so we aim to compute

$$\sum_{i=1}^{47} \frac{1}{r_i}.$$

The reciprocals of the $r_i$ are the solutions to the polynomial obtained by reversing the coefficients of $x^{47} - 1$, so they in fact coincide with the $r_i$. These sum to $\boxed{0}$.

15. Benji has a $2 \times 2$ grid, which he proceeds to place chips on. One by one, he places a chip on one of the unit squares of the grid at random. However, if at any point there is more than one chip on the same square, Benji moves two chips on that square to the two adjacent squares, which he calls a chip-fire. He keeps adding chips until there is an infinite loop of chip-fires. What is the expected number of chips that will be added to the board?

**Answer:** $\frac{17}{4}$

**Solution:** Note that adding just three chips won't create an infinite loop of chip-fires, but it is possible with four chips. In fact, when adding a chip to a board with three chips, adding a chip to any square already occupied by a chip will lead to an infinite loop of chip-fires. There is a $\frac{3}{4}$ probability that the fourth chip will be added to a square with a chip, and a $\frac{1}{4}$ probability that it is added to the only empty square, in which case it will take one more chip to create an infinite chip-fire loop. So, the expected number of chips added is $4(\frac{3}{4}) + 5(\frac{1}{4}) = \boxed{\frac{17}{4}}$.

16. Jason and Valerie agree to meet for game night, which runs from 4:00 PM to 5:00 PM. Jason and Valerie each choose a random time from 4:00 PM to 5:00 PM to show up. If Jason arrives first, he will wait 20 minutes for Valerie before leaving. If Valerie arrives first, she will wait 10 minutes for Jason before leaving. What is the probability that Jason and Valerie successfully meet each other for game night?

**Answer:** $\frac{31}{72}$

**Solution:** Consider a grid where Jason's arrival time is the $x$-axis and Valerie's arrival time is the $y$-axis. We first calculate the complement (probability that Jason and Valerie do not meet).

If Jason arrives first, there is a $\frac{1}{2}(\frac{2}{3}\cdot\frac{2}{3})=\frac{2}{9}$ probability that Valerie does not show up on time. If Valerie arrives first, there is a $\frac{1}{2}(\frac{5}{6}\cdot\frac{5}{6})=\frac{25}{72}$ chance that Jason does not arrive on time. Then, the probability that Jason and Valerie both meet is $1-\frac{2}{9}-\frac{25}{72}=\boxed{\frac{31}{72}}$.

17. Simplify $\sqrt[4]{17+12\sqrt{2}}-\sqrt[4]{17-12\sqrt{2}}$.

**Answer: 2**

**Solution:** We will first show that $\left(3+2\sqrt{2}\right)^2=17+12\sqrt{2}$ and $\left(3-2\sqrt{2}\right)^2=17-12\sqrt{2}$ as follows:

$$\sqrt{17+12\sqrt{2}}=\sqrt{9+8+(2)(3)\left(2\sqrt{2}\right)}=\sqrt{\left(3+2\sqrt{2}\right)^2}$$

$$\sqrt{17-12\sqrt{2}}=\sqrt{9+8+(-2)(3)\left(2\sqrt{2}\right)}=\sqrt{\left(3-2\sqrt{2}\right)^2}$$

In a similar way, we will now show that $(\sqrt{2}+1)^2=3+2\sqrt{2}$ and $(\sqrt{2}-1)^2=3-2\sqrt{2}$:

$$\sqrt{3+2\sqrt{2}}=\sqrt{2+1+(2)(1)\left(\sqrt{2}\right)}=\sqrt{\left(\sqrt{2}+1\right)^2}$$

$$\sqrt{3-2\sqrt{2}}=\sqrt{2+1+(-2)(1)\left(\sqrt{2}\right)}=\sqrt{\left(\sqrt{2}-1\right)^2}$$

Thus, we now have:

$$\sqrt[4]{17+12\sqrt{2}}-\sqrt[4]{17-12\sqrt{2}}=\sqrt{3+2\sqrt{2}}-\sqrt{3-2\sqrt{2}}=\left(\sqrt{2}+1\right)-\left(\sqrt{2}-1\right)=\boxed{2}.$$

18. In quadrilateral $ABCD$, suppose that $\overline{CD}$ is perpendicular to $\overline{BC}$ and $\overline{DA}$. Point $E$ is chosen on segment $\overline{CD}$ such that $\angle AED=\angle BEC$. If $AB=6$, $AD=7$, and $\angle ABC=120^\circ$, compute $AE+EB$.

**Answer: $2\sqrt{37}$**

**Solution:** Since $\angle ABC=120^\circ$, $\angle BAD=60^\circ$, so $BC=4$. Let $A'$ and $B'$ be the reflection of $A$ and $B$ over the segment $\overline{CD}$, respectively. Then note that $AE+EB=AB'$. Letting $F$ be the projection of $B'$ onto $AA'$, we have that $FB'=\sqrt{6^2-3^2}=3\sqrt{3}$, and $FA=A'A-A'F=11$, so $AE+EB=\sqrt{11^2+(3\sqrt{3})^2}=\boxed{2\sqrt{37}}$.

19. How many three-digit numbers $\underline{abc}$ have the property that when it is added to $\underline{cba}$, the number obtained by reversing its digits, the result is a palindrome? (Note that $\underline{cba}$ is not necessarily a three-digit number since before reversing, $c$ may be equal to 0.)

**Answer: 233**

**Solution:** Let our three-digit number be $\underline{abc}=100a+10b+c$. When adding this to $\underline{cba}$, we get

$$(100a+10b+c)+(100c+10b+a)=101(a+c)+20b.$$

Now we casework on the number of digits of this number:

**Case 1: The number has three digits.** Then we must have $a+c\le 9$, and for it to be a palindrome the hundreds digit must equal the units digit, which is $a+c$. Thus the hundreds digit is $a+c$ and we must have $b \leq 4$. Then $20b$ gives us the middle digit of the number, and $101(a+c)$ gives us equal units and hundreds digits, so counting gives us $\binom{10}{2} = 45$ ways to pick $(a, c)$ and 5 ways to pick $b$, for a total of $45 \cdot 5 = 225$.

**Case 2: The number has four digits.** Since $\underline{abc}, \underline{cba}$ are three-digit numbers, their sum cannot exceed $2(999) = 1998$, which means the thousands digit is 1. Since our number is a palindrome then the units digits is 1 as well. The only $(a, c)$ that result in a units digit of 1 is when $a + c = 1, 11$, but $a + c = 1$ can never yield a four-digit number. Thus $a + c = 11$, and then looking at the possible values of $b$ gives a palindrome only when $b = 0$. There are 8 ways to choose $(a, c)$, so this case contributes a total of 8 ways.

Finally, we add up the results from both cases to get $225 + 8 = \boxed{233}$.

20. For some positive integer $n$, $(1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^n = (n^2-1)(1-i)$, where $i = \sqrt{-1}$. Compute the value of $n$.

**Answer: 16**

**Solution:** By finite geometric sums, we can simplify the left hand side to

$$\begin{aligned}(1+i)+(1+i)^2+(1+i)^3+\cdots+(1+i)^n &= (1+i)(1+(1+i)+(1+i)^2+\cdots+(1+i)^{n-1}) \\ &= (1+i)\cdot\frac{(1+i)^n-1}{(1+i)-1} \\ &= \frac{1+i}{i}\cdot((1+i)^n-1) \\ &= (1-i)\cdot((1+i)^n-1).\end{aligned}$$

We can divide by $1-i$ and add 1 on both sides to get $(1-i)\cdot((1+i)^n-1) = (n^2-1)\cdot(1-i) \Rightarrow (1+i)^n = n^2$. Seeing that $n^2$ must be real, $(1+i)^n$ must be real as well, so we then conclude that $n$ is a multiple of 4. Let $n = 4k$ for some postive integer $k$. Then we get

$$\begin{aligned}(1+i)^n &= n^2 \\ (1+i)^{4k} &= 16k^2 \\ (-4)^k &= 16k^2\end{aligned}$$

We see that 4 is the smallest solution to $k$ since $(-4)^4 = 16\cdot4^2 = 256$. After $k = 4$, the magnitude of the left-hand side will increase much faster than that of the right-hand side because the former is exponential while the latter is quadratic. Thus, $k$ must be 4, meaning $n = \boxed{16}$.

21. There exist integers $a$ and $b$ such that $\left(1+\sqrt{2}\right)^{12} = a + b\sqrt{2}$. Compute the remainder when $ab$ is divided by 13.

**Answer: 7**

**Solution:** Let $p = 13$, for brevity. Our first step is to simplify $(1+\sqrt{2})^p$. Using the binomial theorem, we find that

$$\left(1+\sqrt{2}\right)^p = \sum_{k=0}^{p}\binom{p}{k}\left(\sqrt{2}\right)^k.$$

Note that, for each $0 < k < p$, we have that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

has a factor of $p$ in the numerator but not the denominator, so $\binom{p}{k} \equiv 0 \pmod{p}$. So, taking all coefficients $\pmod{p}$, we find that

$$\left(1+\sqrt{2}\right)^p = \sum_{k=0}^{p} \binom{p}{k} \left(\sqrt{2}\right)^k \equiv 1^p + \left(\sqrt{2}\right)^p \pmod{p}.$$

We can compute that $\left(\sqrt{2}\right)^p = 2^{(p-1)/2} \cdot \sqrt{2}$, and we note that $2^{(p-1)/2} = 2^6 \equiv -1 \pmod{13}$. So we find that

$$\left(1+\sqrt{2}\right)^p \equiv 1-\sqrt{2} \pmod{p}.$$

To finish, we expand out

$$\left(a+b\sqrt{2}\right)\left(1+\sqrt{2}\right) = \left(1+1\sqrt{2}\right)^{p-1}\left(1+\sqrt{2}\right) = \left(1+\sqrt{2}\right)^p \equiv 1-\sqrt{2} \pmod{p}.$$

However the left-hand side is $(a+2b)+(a+b)\sqrt{2}$ while the right-hand side is $1-\sqrt{2} \pmod{p}$, which gives the system

$$\begin{cases} a+2b \equiv 1 & \pmod{p}, \\ a+b \equiv -1 & \pmod{p}. \end{cases}$$

This gives $a \equiv -3$ and $b \equiv 2 \pmod{p}$, so in total, we find $ab \equiv -6 \equiv \boxed{7} \pmod{p}$.

22. Austin is at the Lincoln Airport. He wants to take 5 successive flights whose destinations are randomly chosen among Indianapolis, Jackson, Kansas City, Lincoln, and Milwaukee. The origin and destination of each flight may not be the same city, but Austin must arrive back at Lincoln on the last of his flights. Compute the probability that the cities Austin arrives at are all distinct.

**Answer:** $\frac{2}{17}$

**Solution:** First, consider all flight paths where the only restriction is not being able to take a flight to your current location: there are $4^5$ of these paths, because there are 4 destinations to choose from for each flight.

Now, consider the probability of ending up in Lincoln if you start in Lincoln and pick from these 4 destinations randomly every time. We can find this using Markov chains: $p_n$ is the probability of being in Lincoln after $n$ flights. $p_n = (\frac{1}{4})(1-p_{n-1})$, as if you're not in Lincoln on the previous flight you will have a $\frac{1}{4}$ chance of going to Lincoln and if you were in Lincoln then you cannot go to Lincoln. We know that $p_0 = 1$ as we start in Lincoln, so we calculate forward using our answers recursively to get $p_5 = \frac{51}{256}$.

Then we can calculate the probability of a flight path visiting all distinct cities by noticing that we can simply permute the 4 remaining cities as our first 4 destinations, and then Lincoln is guaranteed to be the final destination, giving $4! = 24$ possible flight paths out of a total $4^5$ possible.

Therefore the final probability is

$$\frac{\frac{24}{1024}}{\frac{51}{256}} = \boxed{\frac{2}{17}}.$$

23. Shivani has a single square with vertices labeled $ABCD$. She is able to perform the following transformations:

- She does nothing to the square.

- She rotates the square by 90, 180, or 270 degrees.
- She reflects the square over one of its four lines of symmetry.

For the first three timesteps, Shivani only performs reflections or does nothing. Then for the next three timesteps, she only performs rotations or does nothing. She ends up back in the square's original configuration. Compute the number of distinct ways she could have achieved this.

**Answer: 784**

**Solution:** Note that with these transformation there are only 8 possible configurations, and from a single configuration you can get to every other configuration by doing a transformation. Thus, we only need to determine the number of possible paths for 4 timesteps such that the last timestep can also be a rotation. We note that rotations don't change the clockwise orientation of the vertices $ABCD$. Thus if the last three steps are rotations, then the first three steps must output a rotation. We also note that reflections, no matter when they are performed, flip the orientation of the vertices. Thus the first three transformation must either all be do-nothing transforms or they must consist of two reflections and a do-nothing transform. There are $3 \cdot 4 \cdot 4 + 1 = 49$ ways to do this, since we have three choices for the do-nothing transform and four choices of reflections. We also have the single possible do-nothing transform. For the rotations, the first two we can do whatever we want, so we have 16 possible choices for the first two. The last one is fixed to get back to the original configuration. Thus there are a total of $49 \cdot 16 = \boxed{784}$ ways of doing this.

24. Given that $x$, $y$, and $z$ are a combination of positive integers such that $xyz = 2(x + y + z)$, compute the sum of all possible values of $x + y + z$.

**Answer: 30**

**Solution:** Writing $xyz = 2x + 2y + 2z$, or $x(yz - 2) = 2y + 2z$, we observe that $x = \frac{2y+2z}{yz-2}$, with $yz > 2$. Thus $z \geq 1$ and $y > \frac{2}{z}$.

If $z = 1$, this yields $x = \frac{2y+2}{y-2} = 2 + \frac{6}{y-2}$, so $y = 3, 4, 5, 8$. Respectively, these values produce the triples $(8, 3, 1), (5, 4, 1), (4, 5, 1), (3, 8, 1)$, so the possible values of $x + y + z$ are 10 and 12.

If $z = 2$, then $y \geq 2$, so with $x = \frac{2y+4}{2y-2} = \frac{y+2}{y-1} = 1 + \frac{3}{y-1}$, we get $y = 2, 4$, or $(x, y, z) = (4, 2, 2), (2, 4, 2)$, which both give $x + y + z = 8$.

Finally, if $z \geq 3$, then $y$ can be any positive integer. We have $2y + 2z < yz - 2$ whenever $y(2 - z) < -2z - 2$, or when $y(z - 2) > 2 + 2z$, i.e. $y > \frac{2+2z}{z-2}$. For sufficiently small $z$ ($z \leq 8$), we can manually verify that no triples $(x, y, z)$ exist.

When $\frac{2+2z}{z-2} = 2 + \frac{6}{z-2} < 3$; that is, when $y = 1, 2$ only, we have that $z > 8$. For $y = 1$, we obtain $x = \frac{2+2z}{z-2} = 2 + \frac{6}{z-2}$, which is a contradiction when $z > 8$. For $y = 2$, we similarly obtain $x = \frac{4+2z}{2z-2} = 1 + \frac{6}{2z-2} = 1 + \frac{3}{z-1}$, which also leads to a contradiction when $z > 8$. Hence, we have exhausted all possible triples $(x, y, z)$, so the requested sum is $10 + 12 + 8 = 30$.

*Alternate Solution*

Without loss of generality, suppose $z = \min(x, y, z)$. If $z \geq 3$, we see (1) that $3 \cdot 3 \cdot 3 = 27 > 18 = 2(3 + 3 + 3)$ and (2) that increasing any of $x, y, z$ by 1 increases $xyz$ by at least $3 \cdot 3 = 9$ and increases $2(x + y + z)$ by 2, so $z \geq 3 \implies xyz > 2(x + y + z)$. Thus, we have $z \in \{1, 2\}$.

If $z = 1$, we have:

$$xy = 2x+2y+2 \implies xy-2x-2y+4 = 6 \implies (x-2)(y-2) = 6 \implies (x, y, z) = (8, 3, 1), (5, 4, 1)$$

If $z = 2$, we have:

$$2xy = 2x + 2y + 4 \implies xy - x - y + 1 = 3 \implies (x-1)(y-1) = 3 \implies (x, y, z) = (4, 2, 2)$$

Adding up the possible sums gives $12 + 10 + 8 = \boxed{30}$.

25. Let $\triangle BMT$ be a triangle with $BT = 1$ and height 1. Let $O_0$ be the centroid of $\triangle BMT$, and let $\overline{BO_0}$ and $\overline{TO_0}$ intersect $\overline{MT}$ and $\overline{BM}$ at $B_1$ and $T_1$, respectively. Similarly, let $O_1$ be the centroid of $\triangle B_1MT_1$, and in the same way, denote the centroid of $\triangle B_nMT_n$ by $O_n$, the intersection of $\overline{BO_n}$ with $\overline{MT}$ by $B_{n+1}$, and the intersection of $\overline{TO_n}$ with $\overline{BM}$ by $T_{n+1}$. Compute the area of quadrilateral $MBO_{2021}T$.

**Answer:** $\frac{1}{3 \cdot 2^{2021}}$

**Solution:** By definition of centroid, $BO_n$ and $TO_n$ are medians of $B_nMT_n$ meaning $B_{n+1}$ and $T_{n+1}$ are midpoints of $MB_n$ and $MT_n$ respectively. We can then see that $\frac{MB_{n+1}}{MB_n} = \frac{MT_{n+1}}{MT_n} = \frac{1}{2}$ for all $n$ which means that $\triangle B_{n+1}MT_{n+1} \sim \triangle B_nMT_n$. This means that we also have $\frac{BO_{n+1}}{BO_n} = \frac{1}{2}$ and inducting, we can see that $\frac{BO_{n+1}}{BO_0} = \left(\frac{1}{2}\right)^n$. Furthermore since $O_0$ is the centroid of $\triangle BMT$, $\frac{BO_0}{BK} = \frac{2}{3}$ where $K$ is the midpoint of $BT$. Then, we can see that

$$[MBO_{2021}T] = \frac{BO_{2021}}{BK} \cdot [MBKT] = \frac{BO_{2021}}{BO_0} \cdot \frac{BO_0}{BK} \cdot [MBT] = \frac{1}{2^{2021}} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3 \cdot 2^{2021}}.$$

*Alternate Solution*

First observe that the centroid lies at the mean of the coordinates of the vertices of $\triangle BMT$. In addition, the slope of $\overline{MT}$ is $-2$ and the slope of $BO_0$ is $\frac{2}{3}$, so the intersection with $\overline{MT}$ is its midpoint. Thus, $O_1$ lies $\frac{2}{3}$ up the height of the original triangle. Similarly, we get that $O_n$ is the midpoint of $\overline{O_{n-1}O_{n+1}}$ for all $n$, so $O_n$ is $1 - \frac{2}{3 \cdot 2^n}$ units away from $\overline{BT}$. Hence, the distance from $O_{2021}$ to the base $\overline{BT}$ is $1 - \frac{2}{3 \cdot 2^{2021}}$. This makes the area of $\triangle BO_{2021}T$ equal to $\frac{1}{2} - \frac{1}{3 \cdot 2^{2021}}$, and thus, the area of $MBO_{2021}T$ is $\frac{1}{2}$ minus this, or $\boxed{\frac{1}{3 \cdot 2^{2021}}}$.