1. Towa has a hand of three different red cards and three different black cards. How many ways can Towa pick a set of three cards from her hand that uses at least one card of each color?

   Answer: 18

   Solution: The quickest way is to use complementary counting: using the six cards, Towa has $\binom{6}{3} = 20$ ways to place down a set of three cards. Among these sets, there are two that do not use at least one card of each color, which are all red cards and all black cards. So, there are $20 - 2 = \boxed{18}$ sets in all.

   Alternatively, one can label the cards ABCDEF, where ABC are the red cards and DEF are the black cards, and then list out all of the possibilities.

2. Alice is counting up by fives, starting with the number 3. Meanwhile, Bob is counting down by fours, starting with the number 2021. How many numbers between 3 and 2021, inclusive, are counted by both Alice and Bob?

   Answer: 101

   Solution: We see that 2021 divided by 4 has remainder 1, thus Bob counts all numbers that leave remainder 1 when divided by 4 ($1 \bmod 4$). Alice counts all numbers that leave remainder 3 when divided by 5 ($3 \bmod 5$). To be counted by both Alice and Bob, the number must leave remainder 13 when divided by 20 ($13 \bmod 20$). There are $\boxed{101}$ such numbers, starting from $20 \cdot 0 + 13$ and ending at $20 \cdot 100 + 13$.

3. How many distinct sums can be made from adding together exactly 8 numbers that are chosen from the set $\{1, 4, 7, 10\}$, where each number in the set is chosen at least once? (For example, one possible sum is $1 + 1 + 1 + 4 + 7 + 7 + 10 + 10 = 41$.)

   Answer: 13

   Solution: The small combinations to be summed up allow test-takers to experiment around with the numbers and find all possible values without trepidation. This solution presents a more organized approach. Since each number in the set $\{1, 4, 7, 10\}$ is used at least once, four of the numbers are predetermined to be 1, 4, 7, and 10. So, the problem is equivalent to that of finding the number of distinct sums from adding together 5 numbers, without the restriction that each number is chosen at least once, At the end, we can just add $1 + 4 + 7 + 10 = 22$ to each of these sums to get back to the original problem.

   For this alternate problem, the smallest sum possible is $4 \cdot 1 = 4$, and the largest sum possible is $4 \cdot 10 = 40$. Since each of the numbers in the set $\{1, 4, 7, 10\}$ leave a remainder of 1 when divided by 3, every possible sum must leave the same remainder as 4 does when divided by 3, which is 1. We can increase the sum by 3 at a time by changing a 1 to a 4, a 4 to a 7, or a 7 to a 10, until all numbers in the sum are 10 - this means that every number between 4 and 40 that leaves a remainder of 1 when divided by 3 is attainable. Therefore, the number of possible sums is $\frac{40-4}{3} + 1 = \boxed{13}$.

4. Derek and Julia are two of 64 players at a casual basketball tournament. The players split up into 8 teams of 8 players at random. Each team then randomly selects 2 captains among their players. What is the probability that both Derek and Julia are captains?

   Answer: $\mathbf{\frac{5}{84}}$

   Solution: The probability that both Derek and Julia are captains actually differs based on whether they are on the same team or different teams, so we will need to account for these cases. Fix Derek on a certain team. Then the probability that Julia is also on that team is $\frac{7}{63} = \frac{1}{9}$, since there are only 7 slots left on that team. If they are on the same team, then there is a $\frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28}$ probability that they are both captains. If they are on different teams, then there is a $\frac{2}{8} \cdot \frac{2}{8} = \frac{1}{16}$ probability that they are both captains. The final probability that they are both captains is $\frac{1}{9} \cdot \frac{1}{28} + \frac{8}{9} \cdot \frac{1}{16} = \boxed{\frac{5}{84}}$.

5. How many three-digit numbers $\underline{abc}$ have the property that when it is added to $\underline{cba}$, the number obtained by reversing its digits, the result is a palindrome? (Note that $\underline{cba}$ is not necessarily a three-digit number since before reversing, c may be equal to 0.)

Answer: 233

Solution: Let our three-digit number be $\underline{abc} = 100a + 10b + c$. When adding this to $\underline{cba}$, we get

$$(100a + 10b + c) + (100c + 10b + a) = 101(a + c) + 20b.$$

Now we casework on the number of digits of this number:

Case 1: The number has three digits. Then we must have $a + c \leq 9$, and for it to be a palindrome the hundreds digit must equal the units digit, which is $a + c$. Thus the hundreds digit is $a + c$ and we must have $b \leq 4$. Then $20b$ gives us the middle digit of the number, and $101(a + c)$ gives us equal units and hundreds digits, so counting gives us $\binom{10}{2} = 45$ ways to pick (a, c) and 5 ways to pick b, for a total of $45 \cdot 5 = 225$.

Case 2: The number has four digits. Since $\underline{abc}, \underline{cba}$ are three-digit numbers, their sum cannot exceed $2(999) = 1998$, which means the thousands digit is 1. Since our number is a palindrome then the units digits is 1 as well. The only (a, c) that result in a units digit of 1 is when $a + c = 1, 11$, but $a + c = 1$ can never yield a four-digit number. Thus $a + c = 11$, and then looking at the possible values of b gives a palindrome only when $b = 0$. There are 8 ways to choose (a, c), so this case contributes a total of 8 ways.

Finally, we add up the results from both cases to get $225 + 8 = \boxed{233}$.

6. Compute the sum of all positive integers n such that n^n has 325 positive integer divisors. (For example, $4^4 = 256$ has 9 positive integer divisors: 1, 2, 4, 8, 16, 32, 64, 128, 256.)

Answer: 93

Solution: Observe that $n \neq 1$, so let the prime factorization of n be $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ where the e_i are in increasing order. We have the number of divisors of $n^n = p_1^{e_1 n} p_2^{e_2 n} \dots p_k^{e_k n}$ is

$$325 = (e_1 n + 1)(e_2 n + 1) \dots (e_k n + 1). \ (*)$$

As a result, we look at how to write $325 = 5^2 \cdot 13$ as the product of integers greater than 1. We have 325 can be expressed as 325, $13 \cdot 25$, $5 \cdot 65$, $5 \cdot 5 \cdot 13$.

If the sole term on the RHS of $(*)$ is 324, then $325 = e_1 n + 1$ and $324 = e_1 n$ and n has only one prime factor. We have $324 = 2^2 3^4$, so n is either a power of 2 or a power of 3.

If $n = 2^x$, then $x 2^x = 2^2 3^4$. We have the RHS of this equation is not divisible by 8, so $2^x \leq 4$ and $x \leq 2$. It is easy to check that $x = 1$ and $x = 2$ do not work.

If $n = 3^y$, then $y 3^y = 2^2 3^4$. We have the RHS of this equation is not divisible by 3^5, so $3^y \leq 3^5$ and $y \leq 5$. Of these y, only $y = 4$ satisfies the equation, giving us a solution of $n = 3^4 = 81$.

If the terms on the RHS of $(*)$ are $13 \cdot 25$, then $13 = e_1 n + 1$ and $25 = e_2 n + 1$ and n has two prime factors. We have $e_1 n = 12$ and n has two prime factors, so either $n = 6$ or $n = 12$. We find $n = 6$ fails but $n = 12$ yields a solution.

If the terms on the RHS of $(*)$ are $5 \cdot 65$, then $5 = e_1 n + 1$ and n has two prime factors. However, as $e_1 n = 4$ and n divides 4, this is a contradiction as n can only have one prime factor, 2. Thus, there are no solutions in this case.

If the terms on the RHS of $(*)$ are $5 \cdot 5 \cdot 13$, then $5 = e_1 n + 1$ and n has three prime factors. However, as $e_1 n = 4$ and n divides 4, this is a contradiction similar to before. There are again no solutions in this case.

Thus, the sum of all n is $12 + 81 = \boxed{93}$.

7. For a given positive integer n, you may perform a series of steps. At each step, you may apply an operation: you may increase your number by one, or if your number is divisible by 2, you may divide your number by 2. Let $\ell(n)$ be the minimum number of operations needed to transform the number n to 1 (for example, $\ell(1) = 0$ and $\ell(7) = 4$). How many positive integers n are there such that $\ell(n) \leq 12$?

   Answer: 377

   Solution: Given a positive integer n, we consider the sequence of operations defined using the following greedy algorithm, as follows.

   (a) If $n = 1$, we finish.
   (b) If n is even, we apply the operation $n \mapsto n/2$.
   (c) If n is odd, we apply the operation $n \mapsto n + 1$.

   The key claim is that the greedy algorithm defined above requires length $\ell(n)$ in order to reduce n to 1. In other words, we claim that the greedy algorithm is optimal to reduce n to 1.

   Technically we ought to show that the greedy algorithm always terminates. Well, if n is even, then the greedy algorithm causes n to strictly decrease; if $n = 1$, then we finish automatically; and if $n \geq 3$ is odd, then we add one and will divide by two in the next step, so we see $n \mapsto \frac{n+1}{2}$ still causes n to strictly decrease. So indeed, the algorithm will terminate.

   We now show that the greedy algorithm is optimal, by induction on n. It is optimal for $n = 1$ because $\ell(1) = 0$. Otherwise, $n > 1$, and we have two cases.

   - If n is odd, then the only operation we are allowed to apply is $n \mapsto n + 1$, so the greedy algorithm behaves optimally here.
   - If n is even, then we need to show dividing by two will reduce to 1 faster than adding 1. Well, there is some optimal sequence of operations which reduces n to 1. Suppose that such a sequence of operations starts by adding one $k \geq 0$ times. After these k operations, we divide by 2, so we drop down to $\frac{n+k}{2}$ and continue from there. So it has taken $k + 1$ operations to reduce n to $\frac{n+k}{2}$.
     However, we notice that we have the following optimization: k must be even because n is even, so we could have first divided n by 2 and then added one $k/2$ times, still sending n to $\frac{n}{2} + \frac{k}{2}$. So it has taken $\frac{k}{2} + 1$ operations to reduce n to $\frac{n+k}{2}$.
     But by the optimality of the original sequence, we see that $k + 1 \leq \frac{k}{2} + 1$ is forced. So we must have $k = 0$, meaning that any optimal sequence of operations begins by dividing by 2.

   So in all cases we see that the greedy algorithm does indeed give the optimal sequence of moves, and in fact the greedy algorithm provides the unique such sequence of moves.

We are now ready to finish the problem. Given $m \geq 0$, set a_m to be the number of even positive integers n such that $\ell(n) = m$, and we set b_m to be the number of odd positive integers n such that $\ell(n) = m$. Fixing some $m \geq 1$, we see that each even integer with $\ell(n) = m$ comes from the double of an integer with $\ell(n) = m - 1$, so $a_m = b_{m-1} + a_{m-1}$. Additionally, each odd integer with $\ell(n) = m$ comes from one minus an even integer with $\ell(n) = m$, with the exception of the number 1, which should not be first obtained from subtracting 1 from 2. Noting that $\ell(2) = 1$, we have $b_m = a_{m-1}$ for all $m \geq 3$, and $b_0 = 1$. The system of recurrences is, for all $m \geq 3$, is:

$$\begin{cases} a_m = a_{m-1} + b_{m-1}, \\ b_m = a_{m-1}. \end{cases}$$

We get the recursion $a_m = a_{m-1} + a_{m-2}$ for $m \geq 3$. The first few terms of a_n and b_n are $a_0 = 0$, $b_0 = 1$, $a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$, $a_3 = 1$, and $b_3 = 1$. From this point forward, the recurrence works, and a_n and b_n are the Fibonacci numbers, shifted over by some amount.

The total number of positive integers such that $\ell(n) \leq 12$ is

$$\sum_{m=0}^{12} (a_m + b_m).$$

We can either compute the entire sum by hand or use the fact that the sum of the first k Fibonacci numbers is equal to one less than the $k+2$th Fibonacci number to arrive at the total $\boxed{377}$.

8. Consider the randomly generated base 10 real number $r = 0.\overline{p_0p_1p_2\ldots}$, where each p_i is a digit from 0 to 9, inclusive, generated as follows: p_0 is generated uniformly at random from 0 to 9, inclusive, and for all $i \geq 0$, p_{i+1} is generated uniformly at random from p_i to 9, inclusive. Compute the expected value of r.

Answer: $\mathbf{\frac{10}{19}}$

Solution: By linearity of expectation, we have that

$$E[r] = E[p_0/10] + E[p_1/10^2] + \cdots = \sum_{i=0}^{\infty} \frac{E[p_i]}{10^{i+1}},$$

so it suffices to compute the expected value of each digit. We calculate $E[p_i]$ inductively. First, note that

$$E[p_0] = 0 \cdot \frac{1}{10} + 1 \cdot \frac{1}{10} + \cdots + 9 \cdot \frac{1}{10} = \frac{9}{2}.$$

Now, we try to compute $E[p_{i+1}]$ in terms of $E[p_i]$. Suppose we know that $p_i = d$, where d is a digit from 0 to 9. Then p_{i+1} is randomly generated from d to 9, inclusive, so

$$E[p_{i+1}|p_i = d] = d \cdot \frac{1}{10-d} + (d+1) \cdot \frac{1}{10-d} + \cdots + 9 \cdot \frac{1}{10-d} = \frac{9+d}{2}.$$

Now, to compute the expected value of $E[p_{i+1}]$, we consider all cases for p_i. We have that

$$\begin{aligned} E[p_{i+1}] &= \sum_{d=0}^{9} E[p_{i+1}|p_i = d]P(p_i = d) \\ &= \sum_{d=0}^{9} \frac{9+d}{2} P(p_i = d) \\ &= \sum_{d=0}^{9} \frac{9}{2} P(p_i = d) + \sum_{d=0}^{9} \frac{d}{2} P(p_i = d) \\ &= \frac{9}{2} + \frac{1}{2} E[p_i], \end{aligned}$$

by the law of total probability and definition of expectation.

Thus, we get the recurrence relation

$$\begin{cases} E[p_0] = \frac{9}{2} \\ E[p_{i+1}] = \frac{9}{2} + \frac{1}{2} E[p_i]. \end{cases}$$

Solving this recurrence gives

$$E[p_i] = 9 - \frac{9}{2}\left(\frac{1}{2}\right)^i,$$

so we have that

$$\begin{aligned} E[r] &= \sum_{i=0}^{\infty} \frac{9 - \frac{9}{2}\left(\frac{1}{2}\right)^i}{10^{i+1}} \\ &= \sum_{i=0}^{\infty} \frac{9}{10}\left(\frac{1}{10}\right)^i - \frac{9}{20}\left(\frac{1}{20}\right)^i \\ &= \frac{\frac{9}{10}}{1 - \frac{1}{10}} - \frac{\frac{9}{20}}{1 - \frac{1}{20}} \\ &= \frac{\frac{9}{10}}{\frac{9}{10}} - \frac{\frac{9}{20}}{\frac{19}{20}} \\ &= 1 - \frac{9}{19} \\ &= \boxed{\frac{10}{19}}. \end{aligned}$$

9. Let $p = 101$. The sum

$$\sum_{k=1}^{10} \frac{1}{\binom{p}{k}}$$

can be written as a fraction of the form $\frac{a}{p!}$, where a is a positive integer. Compute $a \pmod p$.

Answer: 5

Solution: Notice that we can remove the fractions immediately by writing

$$a = p! \sum_{k=1}^{10} \frac{1}{\binom{p}{k}} = \sum_{k=1}^{10} \left(p! \cdot \frac{k!(p-k)!}{p!} \right) = \sum_{k=1}^{10} k!(p-k)!.$$

Now, the key observation is that, for given k, we can "flip" $(p-k)!$ around by writing

$$\begin{aligned}(p-k)! &= 1 \cdot 2 \cdot \ldots \cdot (p-k-1) \cdot (p-k) \\ &\equiv -(p-1) \cdot -(p-2) \cdot \ldots \cdot -(k+1) \cdot -k \\ &\equiv (-1)^{p-k} \big((p-1) \cdot (p-2) \cdot \ldots \cdot (k+1) \cdot k \big) \\ &\equiv (-1)^{p-k} \cdot \frac{(p-1)!}{(k-1)!} \pmod{p}.\end{aligned}$$

In particular, we see that

$$\begin{aligned}k!(p-k)! &\equiv k! \cdot (-1)^{p-k} \cdot \frac{(p-1)!}{(k-1)!} \\ &\equiv (-1)^{p-k} k (p-1)! \\ &\equiv (-1)^{p-k+1} k \pmod{p},\end{aligned}$$

where we have used the fact that $(p-1)! \equiv -1 \pmod{p}$ in the last step. We note that $(-1)^{p-k+1} = (-1)^k$, so we see that we are interested in evaluating

$$a \equiv \sum_{k=1}^{10} (-1)^k k \pmod{p}.$$

We can now note that two consecutive terms $2\ell - 1$ and 2ℓ in the sum total to $(-1)^{2\ell-1} \cdot (2\ell - 1) + (-1)^{2\ell} \cdot (2\ell) = 1$, so the total sum evaluates to

$$a \equiv \sum_{\ell=1}^{5} \left((-1)^{2\ell-1} \cdot (2\ell-1) + (-1)^{2\ell} \cdot (2\ell) \right) \equiv \boxed{5} \pmod{p},$$

which is what we wanted.

10. Let N be the number of ways to draw 22 straight edges between 10 labeled points, of which no three are collinear, such that no triangle with vertices among these 10 points is created, and there is at most one edge between any two labeled points. Compute $\frac{N}{9!}$.

Answer: $\mathbf{\frac{23}{24}}$

Solution: We claim that the resulting graph is bipartite and connected.

To prove the graph is bipartite, we would like to show that if the graph has a cycle of length 5, then the graph has less than 22 edges. Suppose S is a set of 5 vertices which is a cycle of length 5, and let T be the set of the other 5 vertices. The subgraph of S contains 5 edges. Since T does not contain a triangle, the subgraph of T has at most 6 edges. Between S and T, no two adjacent vertices from S may connect to the same vertex in T, so for each vertex in T, there is at most 2 edges connecting it to vertices of S, for a total of at most $2 \cdot 5 = 10$ edges. This gives a total of at most 21 edges, which is less than 22. Similar arguments can be made for 7-cycles and 9-cycles, which result in fewer edges. We conclude that the graph is bipartite.

To prove it is connected, we just need to show that any bipartite disconnected graph of 10 vertices has less than 22 edges. We can check that if the connected components have sizes $10-k$ and k, there are at most 20 edges over all values of $k \geq 1$, and the greatest possible number of edges decreases as we split the components further, so the graph is connected.

The facts that the graph is bipartite and connected make counting the number of graphs satisfying this property much easier, as we can just split the vertices into two groups, create the complete bipartite graph, and take away edges until we get to 22. There are two cases.

Case 1: Two groups of 5 vertices. There are $\frac{1}{2}\binom{10}{5}$ ways to split the vertices into two groups of 5. The complete bipartite graph in this case has 25 edges, and we have $\binom{25}{3}$ ways to delete 3 of the edges, for a total of $\frac{1}{2}\binom{10}{5}\binom{25}{3}$ ways in this case.

Case 2: A group of 4 and a group of 6. There are $\binom{10}{4}$ ways to split the vertices into a group of 4 and a group of 6. The complete bipartite graph in this case has 24 edges, and we have $\binom{24}{2}$ ways to delete 2 of the edges, for a total of $\binom{10}{4}\binom{24}{2}$ ways in this case.

The total number of graphs, N, is:

$$\begin{aligned}\frac{1}{2}\binom{10}{5}\binom{25}{3}+\binom{10}{4}\binom{24}{2} &= \binom{10}{4}\binom{24}{2}\left(\frac{1}{2}\cdot\frac{6}{5}\cdot\frac{25}{3}+1\right)\\ &= \frac{10\cdot 9\cdot 8\cdot 7\cdot 24\cdot 23}{4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1}\cdot 6\\ &= 10\cdot 9\cdot 8\cdot 7\cdot 23\cdot 3.\end{aligned}$$

Finally, $\frac{N}{9!} = \boxed{\frac{23}{24}}$.