Time limit: 60 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. Let $g(x) = \int_{2021}^{x} (e^t - 2t)\,\mathrm{d}t$. Compute $g'(2021)$.

2. Let $f(x) = (x+3)(2x+5)(3x+7)(x+1)$. Compute $f^{(4)}(5)$. (Note that $f^{(4)}(5) = f''''(5)$.)

3. A quadratic function in the form $x^2 + cx + d$ has vertex (a, b). If this function and its derivative are graphed on the coordinate plane, then they intersect at exactly one point. Compute b.

4. Compute the area of the region of points satisfying the inequalities $y \le 4 - \frac{x^2}{9}$, $y \ge \frac{x^2}{9} - 4$, $x \le 4 - \frac{y^2}{9}$, and $x \ge \frac{y^2}{9} - 4$.

5. Suppose the following equality holds, where a, b, c are integers and K is the constant of integration:
$$\int \frac{\sin^a(x) - \cos^a(x)}{\sin^b(x)\cos^b(x)}\,\mathrm{d}x = \frac{\csc^c(x)}{c} + \frac{\sec^c(x)}{c} + K.$$
If $a = 2021$, compute $a + b + c$.

6. Let $x_1 = -4$, and for $n \ge 1$, define $x_{n+1} = -4^{x_n}$. Similarly, let $f_1(x) = \sin(\arccos x)$, and for $n \ge 1$, define $f_{n+1}(x) = f_1(f_n(x))$. Compute
$$\lim_{n\to\infty} f_n(2^{x_n}).$$
You may assume that this limit exists.

7. Let $c(x) = \frac{e^x + e^{-2x}}{2}$, defined on the interval $1 \le x \le 2$. Let $c^{-1}(x)$ be the inverse of $c(x)$. Compute
$$\int_{c(1)}^{c(2)} c^{-1}(x)\,\mathrm{d}x\,.$$

8. Define
$$f_n(x) = \int_0^x \frac{t^{6n-1}}{1+t^3}\,\mathrm{d}t$$
for positive integers n and real numbers $0 \le x \le 1$. We can write $f_n(x) = c \cdot \log(p(x)) + h_n(x)$, where $p(x)$ and $h_n(x)$ are polynomials with real coefficients with $p(x)$ monic (coefficient of the highest degree term is 1), and c is a real number. Compute
$$\lim_{n\to\infty} h_n(1).$$

9. Emily plays a game on the real line. Emily starts at the number 1 and starts with 0 points. When she is at the real number a, she chooses a real number b such that $a < b \le 100$. She then moves to b and gains $\frac{4(b-a)}{(a+b)^2}$ points. She repeats this process until she reaches the number 100. Compute the smallest possible value of c such that Emily's score is always less than c.

10. Compute
$$\prod_{n=1}^{\infty} \frac{\pi \arctan(n)}{2\arctan(2n)\arctan(2n-1)}.$$