1. How many permutations of the set $\{B, M, T, 2, 0\}$ do not have B as their first element?

   Answer: 96

   Solution: We have four options for the first letter (the letters other than B), then four options for the second letter, three options for the third, two options for the fourth, and one option for the fifth. Per the multiplication rule, we have $4 \times 4 \times 3 \times 2 \times 1 = \boxed{96}$ such permutations.
   Alternatively, there are $5! = 120$ total permutations of the given set, and one-fifth of them have B as the first element. The answer is then $120 \cdot \frac{4}{5} = \boxed{96}$.

2. Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 3

   Solution: If the first integer is congruent to 1 or $3 \pmod 4$, then the second must be a multiple of 4. This case occurs with probability $\frac{1}{2} \cdot \frac{25}{99} = \frac{25}{198}$. If the first integer is congruent to $2 \pmod 4$, the only requirement for the second integer is that it be even, which occurs with probability $\frac{1}{4} \cdot \frac{49}{99} = \frac{49}{396}$. Finally, if the first integer is a multiple of 4, the product is guaranteed to be a multiple of 4 (probability $\frac{1}{4}$ of this case happening). Hence, the total probability is $\frac{25}{198} + \frac{49}{396} + \frac{1}{4} = \frac{1}{2}$, and our answer is $\boxed{3}$.

3. Compute the remainder when $98!$ is divided by 101.

   Answer: 50

   Solution: Let the remainder be $0 \le x \le 100$. By Wilson's Theorem, $100! \equiv -1 \pmod{101}$, so $100 \cdot 99 \cdot 98! \equiv 100 \cdot 99 \cdot x \equiv -1 \pmod{101}$. Since $100 \equiv -1 \pmod{101}$ and $99 \equiv -2 \pmod{101}$, it remains to solve for x such that $2x \equiv -1 \equiv 100 \pmod{101}$ which gives $\boxed{50}$.

4. Three lights are placed horizontally on a line on the ceiling. All the lights are initially off. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: 82

   Solution: By symmetry, the probability that the rightmost light is on is the same as the probability that the leftmost light is on. The leftmost light changes state if either it or the middle light is switched, which happens with probability $\frac{2}{3}$. The probability that it is on after three seconds is then $\left(\frac{2}{3}\right)^3 + 3 \cdot \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{14}{27}$. Further, the middle light changes state no matter what. Now the expected number of lights on after three seconds, by linearity of expectation, is $\frac{14}{27} + 1 + \frac{14}{27} = \frac{55}{27}$ and our answer is $\boxed{82}$ as desired.

5. Let P be the probability that the product of 2020 real numbers chosen independently and uniformly at random from the interval $[-1, 2]$ is positive. The value of $2P - 1$ can be written in the form $(\frac{m}{n})^b$, where m, n and b are positive integers such that m and n are relatively prime and b is as large as possible. Compute $m + n + b$.

   Answer: 2024

Solution 1: We require that an even number of real numbers are negative (or equivalently, positive), which occurs with probability

$$\frac{1}{3^{2020}} + \binom{2020}{2}\frac{2^2}{3^{2020}} + \cdots + \binom{2020}{2020}\frac{2^{2020}}{3^{2020}}.$$

Observe that the binomial expansion of

$$\left(\frac{1}{3} + \frac{2}{3}\right)^{2020}$$

contains this sum; by symmetry, adding

$$\left(\frac{1}{3} - \frac{2}{3}\right)^{2020}$$

produces twice the desired sum. Simplifying, we get that $P = \frac{1}{2} + \frac{1}{2\cdot 3^{2020}}$, so $2P - 1 = \frac{1}{3^{2020}}$, and our answer is $\boxed{2024}$.

Solution 2: Let P_n be the product after selecting n numbers, and let p_n be the probability that P_n is positive. There are two cases when $P_n > 0$: either $P_{n-1} > 0$ and the nth number is positive, or $P_{n-1} < 0$ and the nth number is negative. This gives

$$p_n = \frac{2}{3} \cdot p_{n-1} + \frac{1}{3} \cdot (1 - p_{n-1}) = \frac{1}{3} \cdot p_{n-1} + \frac{1}{3} \implies 2p_n - 1 = \frac{2}{3} \cdot p_{n-1} - \frac{1}{3} = \frac{1}{3}(2p_{n-1} - 1).$$

Noting that $P_0 = 1$ means that $2p_0 - 1 = 1$, we get $2p_{2020} - 1 = \frac{1}{3^{2020}}$, and our answer is $\boxed{2024}$.

6. Let N be the number of non-empty subsets T of $S = \{1, 2, 3, 4, \ldots, 2020\}$ satisfying $\max(T) > 1000$. Compute the largest integer k such that 3^k divides N.

Answer: 2

Solution: There are 2^{2020} subsets of S, and 2^{1000} subsets of $S' = \{1, 2, \cdots, 1000\}$. The subsets of S' are precisely the subsets of S that *don't* have $\max(T) > 1000$, so we have

$$N = 2^{2020} - 2^{1000} = 2^{1000}(2^{1020} - 1).$$

Now by Euler's theorem (noting that $\varphi(27) = 18$),

$$2^{18} \equiv 1 \pmod{27} \implies 2^{1008} \equiv 1 \pmod{27} \implies 2^{1020} \equiv 4096 \equiv 19 \pmod{27}.$$

It follows that $2^{1020} - 1 \equiv 18 \pmod{27}$, so $k = \boxed{2}$.

7. Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ac = abc + 1$.

Answer: 15

Solution: Without loss of generality, let $a \le b \le c$. We have

$$a + b + c = abc - ab - bc - ca + 1$$

upon subtracting $ab + bc + ca$ from both sides. We then have

$$2a + 2b + 2c = abc - ab - bc - ca + a + b + c + 1$$

upon adding $a+b+c$ to both sides, from which it follows that

$$2a+2b+2c-2 = abc-ab-bc-ca+a+b+c-1 = (a-1)(b-1)(c-1)$$

Then recognize that the LHS is equal to

$$2(a+b+c)-2 = 2((a-1)+(b-1)+(c-1)+2)$$

thereby equpping us to make the substitutions

$$r = a-1, s = b-1, t = c-1 \Rightarrow rst = 2(r+s+t+2) \tag{1}$$

Note that $c = 1$ gives no solution to the original equation, so we know that $t \geq 1$. From here, we observe that

$$t^3 \geq rst \geq 2(1+1+1+2) = 10 \Rightarrow t \geq 3 \tag{2}$$

and also that

$$rst \leq 2(3t+2) \Rightarrow r^2 \leq rs \leq 6 + \frac{4}{t} \leq 7 \Rightarrow r < 3 \Rightarrow r = \{1,2\} \tag{3}$$

If $r = 1$, then

$$st = 2(s+t+3) \Rightarrow (s-2)(t-2) = 10 \Rightarrow (s,t) \in \{(3,12),(4,7)\}.$$

If $r = 2$, then

$$2st = 2(s+t+4) \Rightarrow (s-1)(t-1) = 5 \Rightarrow (s,t) = (2,6).$$

Therefore, $(r,s,t) \in \{(1,3,12),(1,4,7),(2,2,6)\}$. This implies that

$$(a,b,c) \in \{(2,4,13),(2,5,8),(3,3,7)\}.$$

Since there are 3 ordered triples (a,b,c), 2 with 6 permutations and the other with 3 permutations, there are $\boxed{15}$ tuples in total.

8. Dexter is running a pyramid scheme. In Dexter's scheme, he hires *ambassadors* for his company, Lie Ultimate. Any ambassador for his company can recruit up to two more ambassadors (who are not already ambassadors), who can in turn recruit up to two more ambassadors each, and so on (Dexter is a special ambassador that can recruit as many ambassadors as he would like). An ambassador's *downline* consists of the people they recruited directly as well as the downlines of those people. An ambassador earns *executive status* if they recruit two new people and each of those people has at least 70 people in their downline (Dexter is *not* considered an executive). If there are 2020 ambassadors (including Dexter) at Lie Ultimate, what is the maximum number of ambassadors with executive status?

Answer: 27

Solution: Note that we can draw the organization of Dexter's company as a tree, and each person that Dexter recruits is the root of a binary subtree (i.e. a subtree that is binary). We will prove a result about the individual subtrees.
Let e be the number of ambassadors with executive status (in a given subtree). We will prove a general result. Let a be the number of ambassadors (in a binary subtree), and c the cutoff for the downline size of each of someone's recruits in order for that someone to become an executive (standing in for 70 in this case). We claim that

$$e(c+2)+c+1 \leq a$$

(again, in each subtree) when $e > 0$. We prove this claim by induction on e. If $e = 1$, then there is precisely one executive, and they must have recruited two people, each of whom has at least c people in their downline. Then we find that

$$a \geq 1 + 1 + 1 + c + c = 1(c + 2) + c + 1$$

so our desired inequality holds. For our inductive step, suppose we have e executives, where $e > 1$, and suppose that the desired result holds for all smaller values of e. The number of executives is finite, so there is some executive that recruited two non-executives. Consider the binary tree that results from removing this person as well as one of their subtrees (i.e. this person, one of their recruits, and that recruit's entire downline) and viewing their other subtree (their other recruit) as a recruit of the person who recruited them. We have removed at least $1 + 1 + c = c + 2$ people from the organization with this transformation, and we are left with a tree that has one fewer executive. By the inductive hypothesis, we have

$$(e - 1)(c + 2) + c + 1 \leq a - (c + 2)$$

and adding $c + 2$ to both sides gives

$$e(c + 2) + c + 1 \leq a$$

which completes the inductive step, and hence the proof.
Now we can use this result to bound the number of executives at Lie Ultimate. Supposing that we have k subtrees in the overall organization, we can write

$$72e_i + 71 \leq a_i$$

where e_i and a_i are the number of executives and the number of ambassadors in the ith subtree for $i = 1, 2, \cdots, k$. Summing over all i, we find

$$72e + 71k \leq 2019$$

where e is the number of executives at Lie Ultimate (we use 2019 because we ignore Dexter). Evidently, minimizing k will maximize e, so we take $k = 1$ to find that

$$72e \leq 1948 \implies e \leq 27$$

Now the question becomes: is 27 executives attainable? We can draw a 27-node heap and then add two children, each with 70 children of their own, below each leaf, of which there are 14. This creates a tree with

$$1 + 27 + 14 \cdot 2 \cdot 71 = 2016$$

nodes, which is less than 2020. Thus, $\boxed{27}$ executives is attainable and the best that we can do.

9. For any point (x, y) with $0 \leq x < 1$ and $0 \leq y < 1$, Jenny can perform a *shuffle* on that point, which takes the point to $(\{3x + y\}, \{x + 2y\})$ where $\{\alpha\}$ denotes the fractional or decimal part of α (so for example, $\{\pi\} = \pi - 3 = 0.1415...$). How many points p are there such that after 3 *shuffles* on p, p ends up in its original position?

   Answer: 76

   Let a be an integer. Noticing that $\{ax\} = \{a\{x\}\}$ and $\{x + y\} = \{\{x\} + \{y\}\}$, we see that after three shuffles, Jenny takes point (x, y) to point

$$(\{35x + 20y\}, \{20x + 15y\}).$$

We want this to equal to the original point (x, y), or we want

$$(35x + 20y, 20x + 15y) - (x, y) = (34x + 20y, 20x + 14y)$$

to be a lattice point. Looking at what that transformation modulo 1 above does to the entire square $[0, 1) \times [0, 1)$, we want to find all points in the square whose transformations are lattice points. We see that the number of desired points we wish to count is equal to the number of lattice points in the ("half-open") parallelogram generated by the transformation above. Let b be the number of boundary points and i be the number of interior points in the parallelogram. There are only $\frac{b-4}{2}$ noncorner boundary lattice points we want to count (since $(x, 1)$ and $(1, x)$ are technically not part of the square $[0, 1) \times [0, 1)$) plus the one corner point, the origin, that we want to count. We want to count all the interior points, so the total number of lattice points we want to count is

$$i + \frac{b-4}{2} + 1 = i + \frac{b}{2} - 1,$$

which by Pick's theorem is precisely the area of the parallelogram. By the determinant formula for the area of a parallelogram, we get $34 \times 14 - 20 \times 20 = \boxed{76}$. (It's also possible to count the lattice points manually and get $\boxed{76}$, without the use of Pick's Theorem.)

10. Let $\psi(n)$ be the number of integers $0 \le r < n$ such that there exists an integer x that satisfies $x^2 + x \equiv r \pmod{n}$. Find the sum of all distinct prime factors of

$$\sum_{i=0}^{4} \sum_{j=0}^{4} \psi(3^i 5^j).$$

Answer: 54

Solution: By the Fundamental Theorem of Arithmetic, we can find a unique prime factorization of

$$n = \prod_i p_i^{\alpha_i},$$

where k is non-negative, p_i's are odd primes, and α_i's are positive. Then we know

$$x^2 + x \equiv r \pmod{n} \iff (2x + 1)^2 \equiv 4r + 1 \pmod{4n}$$

Hence, $\psi(n)$ is the number of odd residues modulo

$$4n = 2^2 \prod_i p_i^{\alpha_i}$$

between 0 and $4n - 1$, inclusive. Here, we call a a **residue modulo** p if there exists an integer x such that

$$x^2 \equiv a \pmod{p}.$$

Denote the number of residues modulo n between 0 and $n - 1$, inclusive, as $\xi(n)$.

Claim 1. a is a residue modulo $n = \prod_i p_i^{\alpha_i}$, where p_i's are primes and α_i's are positive numbers, if and only if a is a residue modulo $p_i^{\alpha_i}$ for all i.

Proof of Claim 1. If a is a residue modulo

$$n = \prod_i p_i^{\alpha_i},$$

then there exists an integer x, such that

$$x^2 \equiv a \pmod{\prod_i p_i^{\alpha_i}}.$$

For all i, since

$$p_i^{\alpha_i} \mid \prod_i p_i^{\alpha_i},$$

we have

$$x^2 \equiv a \pmod{p_i^{\alpha_i}}$$

and a is a residue modulo $p_i^{\alpha_i}$.

If a is a residue modulo $p_i^{\alpha_i}$ for all i, we can find, for any i, an x_i that satisfies

$$x_i^2 \equiv a \pmod{p_i^{\alpha_i}}.$$

By the Chinese Remainder Theorem, there is a solution

$$x \equiv x' \pmod{\prod_i p_i^{\alpha_i}}$$

to the system of modulo congruences:

$$\forall i, x \equiv x_i \pmod{p_i^{\alpha_i}}.$$

We know for all i,

$$(x')^2 \equiv x_i^2 \equiv a \pmod{p_i^{\alpha_i}}.$$

Then

$$(x')^2 \equiv a \pmod{\prod_i {p_i}^{\alpha_i}}$$

and a is a residue modulo $\prod_i {p_i}^{\alpha_i}$. □

Back to the original problem. From the proof above, we see

$$\psi(n) = \psi(4) \cdot \prod_i \xi(p_i^{\alpha_i}) = \prod_i \xi(p_i^{\alpha_i}).$$

Now, we compute $\xi(p_i^{\alpha_i})$.

Claim 2. $\xi(p_i) = \frac{p_i+1}{2}$.

Proof of Claim 2. Since p_i is an odd prime, we know there is a primitive root g modulo p_i. By definition,

$$\{g, g^2, \ldots, g^{p_i-1} = 1\} = \mathbf{Z}_{p_i}^*.$$

Intuitively, $g^{2k} = (g^k)^2$ are residues modulo p_i for integer $k = 1, \ldots, \frac{p_i-1}{2}$. Including zero yields $\xi(p_i) \geq \frac{p_i+1}{2}$.

Now, we prove $\xi(p_i) \leq \frac{p_i+1}{2}$. Given a nonzero residue a modulo p_i, it must have at least two square roots, as $x^2 \equiv a \pmod{p_i}$ yields $(-x)^2 \equiv a \pmod{p_i}$. Since we only have $p_i - 1$ candidates for square root, we know $\xi(p_i) \leq \frac{p_i-1}{2} + 1 = \frac{p_i+1}{2}$. Hence $\xi(p_i) = \frac{p_i+1}{2}$. □

Back to the original problem. To compute $\xi(p_i^{\alpha_i})$, we prove two lemmas:

Lemma 1 (Hensel's Lemma). Given $\gcd(a, p_i) = 1$, a is a residue modulo $p_i^{\alpha_i}$ if a is a residue modulo p_i.

Proof of Lemma 1. We prove this by induction about α_i.

When $\alpha_i = 1$, this statement is obviously true.

Assume this statement holds when $\alpha_i = k$. Then there exists an integer x_0 such that $x_0^2 \equiv a \pmod{p_i^k}$. Let $x_0^2 \equiv m \cdot p_i^k + a \pmod{p_i^{k+1}}$. Since p_i is an odd prime, there exists an l such that $2x_0 l + m \equiv 0 \pmod{p_i}$. This implies

$$2x_0 l \cdot p_i^k + m \cdot p_i^k \equiv 0 \pmod{p_i^{k+1}},$$

and thus

$$(x_0 + l \cdot p_i^k)^2 \equiv x_0^2 + 2x_0 l \cdot p_i^k + l^2 \cdot p_i^{2k} \equiv m \cdot p_i^k + a + 2x_0 l \cdot p_i^k \equiv a \pmod{p_i^{k+1}}$$

Therefore, a is a residue modulo p_i^{k+1}, and the statement holds. □

Lemma 2. $a = p_i^m a_0$, where $m < \alpha_i$ and $p_i \nmid a_0$, is a residue modulo $p_i^{\alpha_i}$ if and only if m is even and a_0 is a residue modulo p_i.

Proof of Lemma 2. We first prove the "if" direction. By Lemma 1, we know a is a residue modulo $p_i^{\alpha_i}$: there exists an x such that $x^2 \equiv a_0 \pmod{p_i^{\alpha_i}}$. Since

$$(p_i^{\frac{m}{2}} x)^2 \equiv p_i^m a_0 \equiv a \pmod{p_i^{\alpha_i}},$$

we know a is a residue modulo $p_i^{\alpha_i}$.

Now, we prove the "only if" direction. We know there exists an x such that $x^2 \equiv p_i^m a_0 \pmod{p_i^{\alpha_i}}$. Since $p_i^m \mid a$ and $p_i^{m+1} \nmid a$, m must be even. Thus

$$\left(\frac{x}{p_i^{\frac{m}{2}}}\right)^2 \equiv a_0 \pmod{p_i^{\alpha_i - m}}$$

and a_0 is a residue modulo $p_i^{\alpha_i - m}$. Applying Lemma 1, we know a_0 is a residue modulo p_i. □

Back to the original problem. For a given $m = 2k$, $1 \le a_0 \le p_i^{\alpha_i - 2k} - 1$ is coprime with p_i. Each $l \cdot p + 1 \le a_0 \le (l+1) \cdot p - 1$ segments yields, by Lemma 2 and Claim 2, $\frac{p_i - 1}{2}$ residues. There are $p_i^{\alpha_i - 2k - 1}$ such segments, hence $m = 2k$ gives $\frac{p_i - 1}{2} p_i^{\alpha_i - 2k - 1}$ residues modular $p_i^{\alpha_i}$. Summing up all k's and adding 0 as a residue yields

$$\begin{aligned}
\xi(p_i^{\alpha_i}) &= 1 + \sum_{k=0}^{\lfloor \frac{\alpha_i - 1}{2} \rfloor} \frac{p_i - 1}{2} p_i^{\alpha_i - 2k - 1} \\
&= 1 + \frac{p_i - 1}{2} p_i^{\alpha_i - 1} \frac{1 - p_i^{-2 \cdot \lfloor \frac{\alpha_i + 1}{2} \rfloor}}{1 - p_i^{-2}} \\
&= 1 + \frac{p_i^{\alpha_1 + 1}(1 - p_i^{-2 \cdot \lfloor \frac{\alpha_i + 1}{2} \rfloor})}{2(p_i + 1)} \\
&= \begin{cases} 1 + \frac{p_i^{\alpha_i + 1} - 1}{2(p_i + 1)} & \alpha_i = 2l - 1, l \in \mathbb{N}^* \\ 1 + \frac{p_i^{\alpha_i + 1} - p_i}{2(p_i + 1)} & \alpha_i = 2l, l \in \mathbb{N}^* \end{cases}
\end{aligned}$$

Now, with a generic formula, we can rewrite the equation:

$$\begin{aligned}\sum_{i=0}^{4}\sum_{j=0}^{4}\psi(3^i5^j) &= \sum_{i=0}^{4}\sum_{j=0}^{4}\xi(3^i)\cdot\xi(5^j)\\ &= \sum_{i=0}^{4}\xi(3^i)\cdot\sum_{j=0}^{4}\xi(5^j)\\ &= (1+2+4+11+31)\cdot(1+3+11+53+261)\\ &= 7^3\cdot 47.\end{aligned}$$

Therefore, our desired result is $7+47=\boxed{54}$.