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1 Introduction

Tropical geometry is a relatively new area in mathematics. Loosely described, it is a piece-wise linear version of algebraic geometry, over a particular structure known as the *tropical semiring*. Tropical algebraic geometry establishes and studies some general principles to translate problems in algebraic geometry into purely combinatorial ones. This power round will give a brief introduction of some of the basic concepts used in tropical geometry.

2 Tropical Arithmetic

We begin by defining how to do arithmetic. In tropical arithmetic, we use the operators $\oplus$ and $\odot$ in lieu of the usual $+$ and $\times$ in classical arithmetic. These operators are defined on $\mathbb{R} \cup \infty$; that is, the set of real numbers, plus one special "number", ∞, as follows:

$$x \oplus y = \min\{x, y\}$$

$$x \odot y = x + y.$$

For example, the tropical sum $4 \oplus 9 = 4$, and the tropical product $4 \odot 9 = 13$.

1. Compute the following:

   (a) **[1]** $3 \oplus 14$,

   (b) **[1]** $3 \odot 14$,

   (c) **[1]** $2 \oplus \infty$,

   (d) **[1]** $0 \oplus 1$.

   Solution to Problem 1:

   a) 3

   b) 17

   c) 2

   d) 0.

For this power round, we will be working on the set $\mathbb{R} \cup \infty$ equipped with these two operations $\oplus$ and $\odot$. This structure enjoys many of the properties of ordinary arithmetic. For example, these two operations are commutative:

$$x \oplus y = y \oplus x, \text{ and } x \odot y = y \odot x,$$

and associative:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \text{ and } x \odot (y \odot z) = (x \odot y) \odot z.$$

Order of operations is as usual – multiplication comes before addition, just as you learned in elementary school.

2. **[3]** Prove the distributive law:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

Solution to Problem 2:

$$x \odot (y \oplus z) = x + \min(y, z)$$

which is $x \odot y$ if $y < z$ and $x \odot z$ otherwise, which is $(x \odot y) \oplus (x \odot z)$.

The multiplicative identity on this structure is 0; that is to say, $x \odot 0 = x$ for any number x.

3. **[3]** Find the additive identity, which is an element of this set y such that $x \oplus y = x$ for all x.

Solution to Problem 3: This is ∞, because $x \oplus \infty = x$ for all x since ∞ is bigger than any real number.

One feature present in ordinary arithmetic that is missing from tropical arithmetic, however, is subtraction. We cannot define $9 \ominus 4$ meaningfully because $4 \oplus x = 9$ has no solution. However, tropical division *can* be defined, as ordinary subtraction. Satisfying the properties above (commutativity of addition, associativity, additive and multiplicative identities, distributivity, and additive identity being the zero multiplicative element) makes this structure a *semiring*, henceforth called the *tropical semiring*.

3 Tropical Polynomials

We define exponentiation in the usual manner as repeated multiplication, writing, for example, $x^3 = x \odot x \odot x$. In this way, we can have tropical monomials, such as $x_1^2 x_2^{-1} x_3 = x_1 \odot x_1 \oslash x_2 \odot x_3 = x_1 + x_1 - x_2 + x_3$, which are simply linear functions. We note that conversely, any linear function with integer coefficients can be expressed as a tropical monomial.

Tropical polynomials, as you might expect, are then finite tropical sums, or linear combinations, of tropical monomials, and every tropical polynomial in n variables is a function $\mathbb{R}^n \to \mathbb{R}$. Because tropical addition and multiplication are commutative, associative, and distributive, we can multiply tropical polynomials just as we can with ordinary polynomials. For example,

$$(x \oplus 1) \odot (x \oplus -1) = x \odot x \oplus x \odot 1 \oplus x \odot -1 \oplus 1 \odot -1 = x^2 \oplus x \odot -1 \oplus 0.$$

Note that the last equivalence holds because $x \odot -1$ is always less than $x \odot 1$, so the latter term can be removed.

4. **[2]** Compute $(x \oplus 2 \odot y)^2$.

Solution to Problem 4: $x^2 \oplus 4y^2 \oplus 2xy$, by simple distributivity. (Also accept $x^2 \oplus 4y^2$, which is equivalent.)

5. **[4]** Show that the *freshman's dream* holds in tropical arithmetic:

$$(x \oplus y)^n = x^n \oplus y^n$$

for all x, y in the tropical semiring and all integers n.

Solution to Problem 5: $(x \oplus y)^n = x^n \oplus x^{n-1}y \oplus ... \oplus y^n$. If $x < y$, then the minimum of all these terms is x^n, and the minimum is y^n otherwise. Thus, the middle terms are redundant, and the expressions are equivalent.

When evaluating a tropical polynomial in classical arithmetic, what we obtain is the minimum of a finite collection of linear functions.

6. **[6]** Show that such functions $p : \mathbb{R} \to \mathbb{R}$ are *concave*; that is, they satisfy the property that

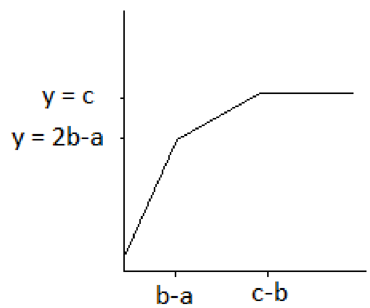
$$p\left(\frac{x_1 + x_2}{2}\right) \geq \frac{p(x_1) + p(x_2)}{2}$$

for all x_1 and x_2. (Note that addition and division in this problem are classical.)

Solution to Problem 6: We note that if such a piecewise linear function has two pieces, it is concave because of a simple calculation of slopes. To eliminate the messiness of n pieces, we can use induction.

4 Graphs and Factors

We begin by examining tropical functions of one variable. For example, consider the second-degree tropical polynomial $y = a \odot x^2 \oplus b \odot x \oplus c$. The graph of this function consists of three lines: $y = a + 2x$, $y = b + x$, and $y = c$, and the value of $p(x)$ at any point is given by the least of the three at that point.



We note that the three lines $y = a + 2x$, $y = b + x$, and $y = c$ intersect at the points $x = b - a$ and $x = c - b$. If $b - a \leq c - b$, then all three lines actually contribute to the graph of our function. We can then break down the polynomial as a tropical product of linear factors by noting that $p(x) = a \odot (x \oplus (b - a)) \odot (x \oplus (c - b))$, and call the points $x = b - a$ and $x = c - b$ the roots of our quadratic equation.

We claim that this is true in general: every single-variable tropical polynomial function can be written uniquely as a tropical product of tropical linear functions. This statement is the *Tropical Fundamental Theorem of Algebra.*

Note that this uniqueness applies to *functions*; distinct tropical polynomials can represent the same function. For example,

$$x^2 \oplus 17 \odot x \oplus 2 = x^2 \oplus 1 \odot x \oplus 2 = (x \oplus 1)^2.$$

The Tropical Fundamental Theorem states that these tropical polynomials can be replaced by an equivalent one that is the tropical product of linear factors.

7. Compute the factorization for the tropical polynomials

   (a) **[2]** $x^2 \oplus 6 \odot x \oplus 28$

(b) **[2]** $x^2 \oplus 14$

(c) **[2]** $x^2 \oplus 4 \odot x \oplus 8$

(d) **[2]** $x^3 \oplus 3 \odot x^2 \oplus 3 \odot x \oplus 1$.

Solution to Problem 7:

(a) $(x \oplus 6) \odot (x \oplus 22)$

(b) $(x \oplus 7)^2$

(c) $(x \oplus 4)^2$

(d) $(x \oplus \frac{1}{3})^3$.

8. (a) **[6]** Find a general rule for factoring the polynomial $a \odot x^2 \oplus b \odot x \oplus c$. Prove that your rule works.

   (b) **[3]** How is this analogous to the quadratic formula in classical arithmetic?

Solution to Problem 8:

(a) If $b - a \leq c - b$, we factor it as $a \odot (x \oplus (b-a)) \odot (x \oplus (c-b))$ as above. Otherwise, the term $b \odot x$ in the expression is redundant, as $a + 2x < b + x$ when $x < b - a$, and $c < b + x$ when $x > c - b$. Thus we can factor it as $(\frac{a}{2} \odot x + \frac{c}{2})^2$.

(b) We can consider the discriminant $d = c - 2b + a$. When $d > 0$, we have two distinct factors, and when $d \leq 0$, we have multiple roots.

9. **[10]** Show that every single-variable tropical polynomial can be factored uniquely into a tropical product of tropical linear functions.

Solution to Problem 9:

Theorem 1.4. (Tropical Fundamental Theorem of Algebra) *Every tropical polynomial function $p(x)$in one variable x can be written uniquely as a product*

$$p(x) \quad = \quad u \odot (x \oplus v_1) \odot (x \oplus v_2) \odot \cdots \odot (x \oplus v_m). \tag{2}$$

The rational numbers $v_1, v_2, \dots, v_m$ are the roots *of $\tilde{p}(x)$ but u is a real number.*

Proof. The tropical polynomial function can be written uniquely as

$$p(x) \quad = \quad d_1 \odot x^{\odot c_1} \oplus d_2 \odot x^{\odot c_2} \oplus \cdots \oplus d_r \odot x^{\odot c_r},$$

where $c_1 > c_2 > \cdots > c_r$ are non-negative integers, $d_1, \dots, d_r$ are arbitrary real numbers, and none of the terms $d_i \odot x^{\odot c_i}$ are redundant in representing the function $p : \mathbb{R} \to RR$. Then the tropical polynomial $p(x)$ equals

$$d_1 \odot \left(x^{c_1-c_2} \oplus (d_2 - d_1)\right) \odot \left(x^{c_2-c_3} \oplus (d_3 - d_2)\right) \odot \cdots \odot \left(x^{c_{r-1}-c_r} \oplus (d_r - d_{r-1})\right) \odot x^{c_r}.$$

Each binomial factor can be factored further into a product of linear forms:

$$d_1 \odot \left(x \oplus \frac{d_2 - d_1}{c_1 - c_2}\right)^{\odot c_1 - c_2} \odot \left(x \oplus \frac{d_2 - d_3}{c_2 - c_3}\right)^{\odot c_2 - c_3} \odot \cdots \odot \left(x \oplus \frac{d_r - d_{r-1}}{c_{r-1} - c_r}\right)^{\odot c_{r-1} - c_r} \odot x^{c_r}$$

This is the desired representation of $p(x)$ as a product of linear terms.

To see that the factorization (2) is unique, we argue as follows. We can recover the constants u and m because $p(x) = mx + u$ for $x \ll 0$. The roots v_i are the places where the graph of $p(x)$ has a breakpoint, and the multiplicity of a root is the difference of the slope to the left minus the slope to the right. □

10. **[3]** Give an example of a tropical polynomial p of two or more variables, and two different factorizations of p into irreducibles, or functions that cannot be factored further. Thus, we note that the Fundamental Theorem of Algebra does not hold for tropical multivariable polynomials.

    Solution to Problem 10: Solutions may vary. One example:

$$(x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0) = (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0).$$

5 Tropical Eigenvalues

Matrix multiplication generalizes tropically as well. We define the product of two tropical matrices A and B similarly to classical matrix multiplication: the entry in the ith row and jth column of AB is the number $a_{i1} \odot b_{1j} \oplus ... \oplus a_{in} \odot b_{nj}$, where a_{ij} and b_{ij} denote the entries of A and B, respectively, on the ith row and jth column. For example,

$$\begin{pmatrix} 3 & 3 \\ 0 & 7 \end{pmatrix} \odot \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}.$$

11. **[2]** Compute the tropical product

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solution to Problem 11:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Let A be an $n \times n$ matrix. An *eigenvalue* of A is a real number λ for which $A \cdot v = \lambda v$ for some $v \in \mathbb{R}^n$; the vector v is then called an *eigenvector* of A. This has a tropical equivalent, as well: an eigenvalue of a tropical matrix A is a number λ for which $A \odot v = \lambda \odot v$ for some vector v.

12. Compute tropical eigenvalues of the matrices

    (a) **[4]**

    $$\begin{pmatrix} \infty & 1 \\ 1 & \infty \end{pmatrix}$$

    (b) **[4]**

    $$\begin{pmatrix} \infty & 1 & 0 \\ 0 & \infty & 1 \\ 1 & 0 & \infty \end{pmatrix}.$$

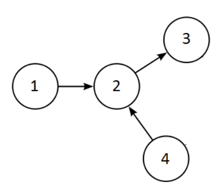
    Solution to Problem 12:

    (a) $\lambda = 2$, $v = (11)$

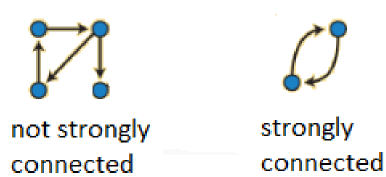
    (b) $\lambda = 1$, $v = (111)$.

13. Let G be a directed graph, and label the vertices of G as $1, \dots, n$. An adjacency matrix of a graph G is a matrix A where a_{ij} is the weight of the edge from vertex i to vertex j, if such an edge exists. If it does not exist, we consider the weight of that edge to be ∞.

    (a) **[2]** Find the adjacency matrix of the following graph. Assume all edges shown have weight 1.

    (b) **[2]** A directed graph is called *strongly connected* if there is an (oriented) path in each direction between each pair of vertices of the graph.

    Is the graph in part (a) strongly connected? Why or why not?

(c) **[10]** If A is the adjacency matrix of a strongly connected directed graph, what can we say about its eigenvalues?

Solution to Problem 13:

(a) $\begin{pmatrix} \infty & 1 & \infty & \infty \\ \infty & \infty & 1 & \infty \\ \infty & \infty & \infty & \infty \\ \infty & 1 & \infty & \infty \end{pmatrix}$.

(b) No, because there is no oriented path from 2 to 1 (for example).

(c) The following theorem:

Theorem 5.1.1. *Let A be a tropical $n \times n$-matrix whose graph $G(A)$ is strongly connected. Then A has precisely one eigenvalue $\lambda(A)$. That eigenvalue equals the minimal normalized length of any directed cycle in $G(A)$.*

Proof. Let $\lambda = \lambda(A)$ be the minimum of the normalized lengths over all directed cycles in $G(A)$. We first prove that $\lambda(A)$ is the only possibility for an eigenvalue. Suppose that $\mathbf{z} \in \mathbb{R}^n$ is any eigenvector of A, and let γ be the corresponding eigenvalue. For any cycle $(i_1, i_2, \ldots, i_k, i_1)$ in $G(A)$ we have

$$a_{i_1 i_2} + z_{i_2} \geq \gamma + z_{i_1}, \; a_{i_2 i_3} + z_{i_3} \geq \gamma + z_{i_2},$$
$$a_{i_3 i_4} + z_{i_4} \geq \gamma + z_{i_3}, \; \ldots, \; a_{i_k i_1} + z_{i_1} \geq \gamma + z_{i_k}.$$

Adding the left hand sides and the right hand sides, we find that the normalized length of the cycle is greater than or equal to γ. In particular, we have $\lambda(A) \geq \gamma$. For the reverse inequality, start with any index i_1. Since $\mathbf{z}$ is an eigenvector with eigenvalue γ, there exists i_2 such that $a_{i_1 i_2} + z_{i_2} = \gamma + z_{i_1}$. Likewise, there exists i_3 such that $a_{i_2 i_3} + z_{i_3} = \gamma + z_{i_2}$. We continue in this manner until we reach an index i_l which was already in the sequence, say, $i_k = i_l$ for $k < l$. By adding the equations along this cycle, we find that

$$(a_{i_k,i_{k+1}} + z_{i_{k+1}}) + (a_{i_{k+1},i_{k+2}} + z_{i_{k+2}}) + \cdots + (a_{i_{l-1},i_l} + z_{i_l})$$
$$= (\gamma + z_{i_k}) + (\gamma + z_{i_{k+1}}) + \cdots + (\gamma + z_{i_l}).$$

We conclude that the normalized length of the cycle $(i_k, i_{k+1}, \ldots, i_l = i_k)$ in $G(A)$ is equal to γ. In particular, $\gamma \geq \lambda(A)$. This proves that $\gamma = \lambda(A)$.

It remains to prove the existence of an eigenvector. Let B be the matrix obtained from A by (classically) subtracting $\lambda(A)$ from every entry in A. All cycles in the weighted graph $G(B)$ have non-negative length, and there exists a cycle of length zero. Using tropical matrix operations we define

$$B^* = B \oplus B^2 \oplus B^3 \oplus \cdots \oplus B^n.$$

The entry B^*_{ij} in row i and column j of the matrix B^* is the length of a shortest path from node i to node j in the weighted directed graph $G(B)$. Since the graph is strongly connected, we have $B^*_{ij} < \infty$. Moreover,

$$(\mathrm{Id} \oplus B) \odot B^* = B^*. \qquad (5.1.2)$$

Here $\mathrm{Id} = B^0$ is the tropical identity matrix whose diagonal entries are 0 and off-diagonal entries are ∞. Fix any node j that lies on a zero length cycle of $G(B)$, and let $\mathbf{x} = B^*_{\cdot j}$ denote the jth column vector of the matrix B^*. We have $x_j = B^*_{jj} = 0$. This property together with (5.1.2) implies

$$\mathbf{x} = (\mathrm{Id} \oplus B) \odot \mathbf{x} = \mathbf{x} \oplus B \odot \mathbf{x} = B \odot \mathbf{x},$$

and we conclude that $\mathbf{x}$ is an eigenvector with eigenvalue λ of our matrix A:

$$A \odot \mathbf{x} = (\lambda \odot B) \odot \mathbf{x} = \lambda \odot (B \odot \mathbf{x}) = \lambda \odot \mathbf{x}.$$

This completes the proof of Theorem 5.1.1. □

6 Bonus: We Bet You Want To Know What's On The T-Shirt

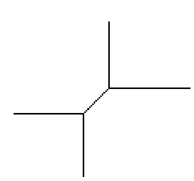
We have defined a tropical polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ as the minimum of a finite set of linear functions. For each such p, we now define the *hypersurface* $V(p)$ to be the set of all points in $\mathbb{R}^n$ for which this minimum is attained at least twice.

For instance, if $p(x) = a \odot x^2 \oplus b \odot x \oplus c$ and $b - a < c - b$, then the hypersurface consists of the two points $x = b - a$ and $x = c - b$, the roots of the polynomial.

14. **[4]** Graph the hypersurface for the polynomial in two variables

$$p(x, y) = -1 \odot x \odot y \oplus x \oplus y \oplus 0.$$

Solution to Problem 14:



7 Conclusion

Tropical geometry has many applications. It lies at the interface between algebraic geometry and combinatorics, with connections to many other areas. It has uses in linear optimization and dynamic programming, such as in job scheduling, location analysis, or finding shortest paths of a weighted directed graph. It is used to solve integer programs, for statistical inference, and in at least one case, to design auctions. Classical objects (lines, polynomials, and curves) can be transformed into tropical objects while preserving many of their characteristics, as well as vice versa, providing a new way of analyzing structures in algebraic geometry and simplifying the construction of curves with desired properties.

8 Trivia

Tropical geometry was initially known as max-plus geometry (from a related strain that takes the convention of $x \oplus y = \max(x, y)$). French mathematicians settled on the term "tropical geometry" to honor their colleague Imre Simon, who worked on the subject in Brazil.