Power Round

Welcome to the power round! This year's topic is the theory of orthogonal polynomials.

I. You should order your papers with the answer sheet on top, and you should number papers addressing the same question. Include your **Team ID** at the top of each page you submit.

II. You may reference anything stated or cited earlier in the test, even if you do not understand it. You may not reference outside sources or proofs to answers not given on the same page.

III. You have **60 minutes** to answer 16 questions, cumulatively worth **100 points**. Good luck!

0 Introduction

As far as I am concerned, the primary purpose of a power round is to show you, the students, what it is like to do mathematics at the university level. I take my responsibility as a representative of higher education very seriously, and I hope you will enjoy some "Eureka" moments during this test. I encourage you most of all to read everything, for even unsolved problems may be understood later.

I have included a series of conceptual questions on this test that account for approximately one quarter of all possible points. You should try to figure out the motivation behind the question and review the recent material to determine how best to answer. If a problem does not explicitly require demonstration of a proof or computation, you may optionally choose to supplement your answer with either. However, if you see terms such as *prove*, *verify*, or *determine*, proof techniques are required for full points. Problems with spots provided on the answer sheet require no explanation.

Think critically. A mathematician always knows exactly what she is talking about, and you may try to do the same by paying careful attention to the definitions. A good beginning is your best way to partial credit on the harder questions, so make sure you do know what you are talking about.

The set of **natural numbers** is $\{0, 1, 2, \ldots\}$. A natural (number) n is an element of this set.
The set of **real numbers** is $\mathbb{R}$. A real number c is a number found on the number line.

A **polynomial** is a function p, from the real numbers to the real numbers, that has the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ for real numbers $a_n, a_{n-1}, \ldots, a_0$ (where $a_n \neq 0$ or $n = 0$) and natural n. The **degree** of the polynomial is n. A **root** of the polynomial is a value c such that $p(c) = 0$. The **Fundamental Theorem of Algebra** states that polynomials of degree n have at most n roots. By convention, this test only contains polynomials with x as the parameter.

One term I use but do not define is *function space*. Formally, I mean a vector space (over the real numbers) whose elements are functions sharing domain and codomain. Informally, I mean a set of real-valued functions you can add together. Some examples of function spaces are the set of real numbers $\mathbb{R}$ (i.e., the collection of constant functions from $\mathbb{R}$ to $\mathbb{R}$), the set of polynomials $\mathcal{P}$, and the set of functions f_a^b of the form $f_a^b(x) = a \cdot e^x + b \cdot \sqrt[3]{x}$ for some real numbers a and b.

1 Functionals (24 pts)

All areas of mathematics have some concept of *mapping* crucial to development of the theory. A **function** maps elements from one value to another. The **domain** is the set of values on which the function is defined, and the **codomain** is the set of possible values into which the function maps. The **range** is the precise set of values the function can achieve. The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2 + \sqrt{x} + 1$ has domain $[0, \infty)$, codomain $\mathbb{R}$, and range $[1, \infty)$. Henceforth, we will speak only of real-valued functions— functions whose codomain is the set of real numbers.

- A **functional** maps (real-valued) functions to real numbers. Its domain is a function space, and its codomain is the set of real numbers.
- A **linear** functional is a functional $\mathbf{T}$ satisfying:
  1. For all functions f and g in its domain, $\mathbf{T}(f + g) = \mathbf{T}(f) + \mathbf{T}(g)$.
  2. For all real numbers c and functions f in its domain, $c \cdot \mathbf{T}(f) = \mathbf{T}(c \cdot f)$.

(The term function space is defined in the introduction.)

For instance, a functional $\mathbf{Lead} : \mathcal{P} \to \mathbb{R}$ may take a polynomial as input and return the coefficient of the term of highest degree, mapping $(2x^2 + 3x + 1) \mapsto 2$ and $(2x^2 + 3x^3 + 1) \mapsto 3$.

1. The following functionals are linear. Use the properties of linearity to determine the answers.
   (a) [**1**] If $\mathbf{A}(1) = 2$ and $\mathbf{A}(x) = 3$, find $\mathbf{A}(2x - 1)$.
   (b) [**1**] If $\mathbf{B}(2x + 1) = 0$, $\mathbf{B}(2x^2 - 4x + 6) = 6$, and $\mathbf{B}(x^3 + 7x) = 8$, find $\mathbf{B}(x^3 + x^2 + x + 1)$.
   (c) [**1**] If $\mathbf{C}(\cos x) = 4$ and $\mathbf{C}(\sin x) = 2\sqrt{3}$, find $\mathbf{C}(\sin(x + \frac{\pi}{3}))$.
   (d) [**1**] If $\mathbf{D}(x^k) = 2k - 1$ for all natural k, find $\mathbf{D}((2x - 1)^{10})$.

2. For each of the following, prove the statement or provide a counterexample.
   (a) [**2**] Every (real-valued) function is a functional.
   (b) [**2**] Every functional is a (real-valued) function.
   (c) [**2**] The codomain of a functional is always contained in its domain.

3. Answer each of the following questions in a clear and concise manner.
   (a) [**3**] Why is it conventional to use codomain instead of range when defining a function?
   (b) [**3**] Is there such a thing as an "inverse functional"? For instance, can you construct a mapping $\Lambda : \mathbb{R} \to \mathcal{P}$ that is an inverse to **Lead**? What limitations, if any, are there?
   (c) [**4**] Consider the functional $\mathbf{Ev}_2(\varphi) = \varphi(2)$ and the function $f(x) = xy + z$, where y and z are fixed constants. Explain the difference in meaning between $\mathbf{Ev}_2(f)$ and $\mathbf{Ev}_2(xy + z)$. Now, consider the function $g(x) = x^2 + x$. Explain the difference in meaning between $\mathbf{Ev}_2(g)$ and $\mathbf{Ev}_2(x^2 + x)$. Is either mistake acceptable? Is it possible to avoid this type of mistake without separately defining a function, as done here?

We will now restrict our attention to $\mathcal{P}$, the function space of polynomial functions. For any real numbers $a < b$, we define a linear functional $\mathbf{Int}_{[a,b]}$ with domain $\mathcal{P}$ as follows.

$$\text{For any natural } n,\ \mathbf{Int}_{[a,b]}(x^n) = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

Using the definition of linearity, we may extend this definition to all polynomials. For instance,

$$\begin{aligned}\mathbf{Int}_{[0,1]}(6x^2 + 2x + 7) &= 6 \cdot \mathbf{Int}_{[0,1]}(x^2) + 2 \cdot \mathbf{Int}_{[0,1]}(x) + 7 \cdot \mathbf{Int}_{[0,1]}(1) \\ &= 6 \cdot \left(\frac{1^3 - 0^3}{3}\right) + 2 \cdot \left(\frac{1^2 - 0^2}{2}\right) + 7 \cdot \left(\frac{1^1 - 0^1}{1}\right) \\ &= 2 + 1 + 7 = 10.\end{aligned}$$

In fact, the definition for $\mathbf{Int}_{[a,b]}$ can be extended to functions beyond just polynomials. A purely formulaic reason for this is that *well-behaved* functions can be approximated *very well* by polynomial functions. But for this test, it is only necessary to know how to apply the functional to polynomials.

4. (a) **[1]** Evaluate $\mathbf{Int}_{[0,2]}(3x^2 + 2x)$.

   (b) **[1]** Evaluate $\mathbf{Int}_{[1,3]}(x^7 + x^3)$.

   (c) **[2]** Prove that for any polynomial p and real numbers $a < b < c$,

$$\mathbf{Int}_{[a,b]}(p) + \mathbf{Int}_{[b,c]}(p) = \mathbf{Int}_{[a,c]}(p).$$

It may be useful later to note that for any real numbers $a < b$ and polynomial p, there exists a value c satisfying $a < c < b$ such that $\mathbf{Int}_{[a,b]}(p) = (b - a) \cdot f(c)$. This is the **Mean Value Theorem**.

2 Simple Orthogonality (25 pts)

Two polynomials f and g are ***simply* orthogonal** if

$$\mathbf{Int}_{[-1,1]}(f \cdot g) = 0.$$

A set of polynomials is simply orthogonal if any distinct two of its elements are simply orthogonal.

5. Which of the following sets are simply orthogonal?
   (a) **[2]** $\{1, x\}$; $\{1, x^2\}$; $\{x, x^2\}$; $\{1, x^2 - \frac{1}{3}\}$; $\{x^2 - 1, x^3\}$
   (b) **[2]** $\{1, x, x^2\}$; $\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$; $\{1, x^2 - 1, x^3\}$; $\{1, x^2 - \frac{1}{3}, x^3 + 2x\}$

6. For each of the following, find a nonzero function satisfying the given condition, or prove none exist.
   (a) **[2]** Find a linear polynomial simply orthogonal to each of 1, $x^2 - \frac{1}{3}$, and $x^3 + 2x$.
   (b) **[2]** Find a cubic polynomial simply orthogonal to each of 1, x, and $x^2 - \frac{1}{3}$.

A function f is **symmetric** if $f(c) = f(-c)$ for all c in its domain. A function f is **antisymmetric** if $f(c) = -f(-c)$ for all c in its domain. For polynomials, the domain is all real numbers.

7. Answer the following questions on symmetric and antisymmetric polynomials.
   (a) **[2]** Prove that if a polynomial is symmetric, then each of its terms has even degree.
   (b) **[2]** Prove that if a polynomial is antisymmetric, then each of its terms has odd degree.
   (c) **[1]** Prove, for any antisymmetric polynomial p, that $\mathbf{Int}_{[-1,1]}(p) = 0$.

The **Legendre polynomials** comprise a sequence of simply orthogonal polynomials, the n^{th} of which is degree n. Any one Legendre polynomial is orthogonal to any other Legendre polynomial. They begin $P_0(x) = 1$ and $P_1(x) = x$ and are subject to the standardization $P_n(1) = 1$ for all natural n. They are uniquely determined by this definition, but an equivalent definition is

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) x P_n(x) - \left(\frac{n}{n+1}\right) P_{n-1}(x) \text{ for all natural } n > 0.$$

8. (a) **[2]** Compute $P_2(x)$, $P_3(x)$ and $P_4(x)$.
   (b) **[2]** Prove that all terms of a Legendre polynomial have the same parity of degree.
   (c) **[2]** Verify with computation or proof that P_4 is orthogonal to P_0, P_1, P_2, and P_3.

9. (a) **[2]** Express x^3 as a sum of distinct nonzero multiples of Legendre polynomials.
   (b) **[4]** The team across the room got a different answer for part (a). Prove them wrong.

For more on Legendre polynomials, go directly to Section 4. For more on orthogonal polynomials in general, continue to Section 3.

3 Orthogonal Polynomials (28 pts)

More generally, orthogonal polynomials arise in a space equipped with an **inner product**. Inner products are two-variable functions that are linear functionals in either variable. In particular, we are concerned with inner products of the following form, where a and b are real numbers and w is a polynomial that is positive throughout the interval $(a,\, b)$.

$$\langle p,\, q\rangle = \mathbf{Int}_{[a,\, b]}(p \cdot q \cdot w) \text{ for all polynomials } p \text{ and } q.$$

Two polynomials p and q are **orthogonal** if $\langle p,\, q\rangle = 0$. As with Legendre polynomials, we may construct sequences of orthogonal polynomials. Henceforth, the term **orthogonal polynomials** refers to a sequence of polynomials orthogonal to each other, the n^{th} of which is degree n. The **Gram-Schmidt process** provides an immediate method of creating some orthogonal polynomials:

$$p_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n,\, p_k\rangle}{\langle p_k,\, p_k\rangle} p_k(x) \text{ for all natural } n.$$

10. (a) **[3]** Prove that the Gram-Schmidt process does produce orthogonal polynomials.
    (b) **[3]** Prove that any polynomial may be expressed as a sum of distinct nonzero multiples of orthogonal polynomials in precisely one way.

11. Let n be a nonzero natural number.
    (a) **[2]** Prove that $\langle p_n,\, q\rangle = 0$ for all polynomials q of degree less than n.
    (b) **[2]** Prove that if p is a nonzero polynomial that is nonnegative throughout the interval $(a,\, b)$, then $\mathbf{Int}_{[a,\, b]}(p) > 0$. (Refer to the end of Section 1 for a relevant theorem.)
    (c) **[4]** Prove that $p_n(x)$ has precisely n distinct real roots in the interval $(a,\, b)$.

In the study of orthogonal polynomials, two values assist in characterizing the relation between different elements of the sequence. We define sequences of these values in the following way. (Recall the functional **Lead** from Section 1 that returns the leading coefficient of a polynomial.)

$$k_n = \mathbf{Lead}(p_n) \text{ and } h_n = \langle p_n,\, p_n\rangle.$$

12. Let $\{p_0,\, p_1,\, p_2,\, \ldots\}$ be orthogonal polynomials.
    (a) **[4]** Prove, for some natural $n > 0$ and real numbers a_n and b_n independent of x, that
    $$p_{n+1}(x) - \frac{k_{n+1}}{k_n} \cdot x p_n(x) = a_n p_n(x) + b_n p_{n-1}(x).$$
    (b) **[2]** Determine the value of b_n in terms of h_{n+1}, k_{n+1}, h_n, k_n, h_{n-1}, and k_{n-1}.

13. Answer each of the following questions in a clear and concise manner.
    (a) **[4]** Is there a choice of a, b, and polynomial w that would make the sequence of polynomials $\{1,\, x,\, x^2,\, x^3,\, \ldots\}$ orthogonal with respect to the inner product described above? Why or why not?
    (b) **[4]** What can you say about the relationship between the roots of two distinct orthogonal polynomials? A false response will get 0 points, while a true response will receive points in proportion to the strength of its implications.

4 Legendre Polynomials (23 pts)

One reason orthogonal polynomials are useful is that they are very good at approximating other functions. In particular, they provide a solution to the *least squares* problem for function approximation. Note our inner product is

$$\langle p,\, q\rangle = \mathbf{Int}_{[-1,\,1]}(p \cdot q) \text{ for all polynomials } p \text{ and } q.$$

Then, a **Legendre approximation** p of degree n (natural n) to the polynomial q is

$$p(x) = \sum_{k=0}^{n} \frac{\langle q,\, P_k\rangle}{\langle P_k,\, P_k\rangle} P_k(x).$$

14. (a) **[2]** Find a Legendre approximation of degree 2 to the function $f_1(x) = x^4$.
    (b) **[2]** Find a Legendre approximation of degree 3 to the function $f_2(x) = x^5$.
    (c) **[4]** Find with proof a general form for the value $\langle P_k,\, P_k\rangle$ in terms of natural k.

15. Draw a graph of a function and one of its Legendre approximations.
    (a) **[4]** Where is the Legendre approximation a close approximation to a function? What value(s) in particular is (are) minimized by the Legendre approximation?
    (b) **[3]** Prove that the Legendre approximation of degree n of a polynomial of degree n is the polynomial itself.

Just as Legendre polynomials may be defined using a recurrence relation, they may be defined with a series relation as well. For sufficiently small t (say, $|t| < \frac{1}{2x}$), the following holds for all x.

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}.$$

16. (a) **[3]** Verify with proof the equivalence for $x = 1$ and $x = -1$.
    (b) **[5]** Prove the following identity. For all natural n,

$$\frac{\sin((n+1)x)}{\sin x} = \sum_{k=0}^{n} P_k(\cos x)P_{n-k}(\cos x).$$