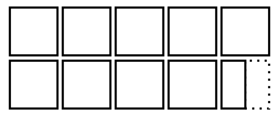
1. What is the value of $16^2 - 11^2 + 6^2 - 1^2$?

   Answer: 170

   Solution: We can compute this directly by calculating the values of the squares and subtracting:

$$16^2 - 11^2 + 6^2 - 1^2 = 256 - 121 + 36 - 1 = \boxed{170}.$$

2. If Aedan bakes 25 identical brownie pieces, and Brian eats 9 and a half of those pieces, what percentage of the brownies did Brian eat? **If the answer is x%, write only x as your answer.**

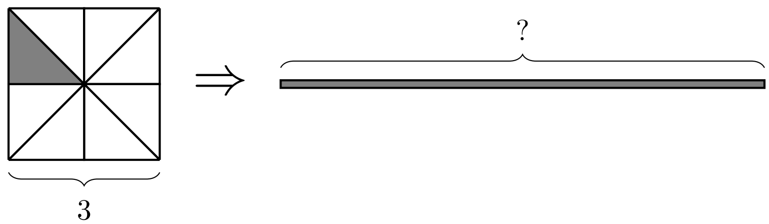


   Answer: 38

   Solution: Brian eats 9 + 1/2 pieces, so the percentage of the total he eats is

$$\frac{9.5}{25} = \frac{9.5 \times 4}{100} = \boxed{38}\%.$$

3. A square shaped pizza dough with side length 3 is divided into 8 slices of equal area. One slice is removed and molded (preserving the area) into a long rectangular pizza dough with one side of length $\frac{1}{8}$. What is the length of the longer side?



   Answer: 9

   Solution: The area of the pizza is $3^2 = 9$. We divide it into eight slices of equal area, so each slice has an area of $\frac{9}{8}$.

   Since this area gets molded into a new rectangle with one side of length $\frac{1}{8}$, the other side must be of length $\boxed{9}$.

4. Jeslyn writes down five numbers whose arithmetic average is 6. The first two numbers are 3 and 7, the third number is half the fifth number, and the fourth number is equal to the fifth number. What was the fifth number that Jeslyn wrote down?

   Answer: 8

   Solution: Let x be the third number. The third number is half the fifth number, so the fifth number is $2x$. The fourth number is equal to the fifth number, so the fourth number is $2x$.

$$3 + 7 + x + 2x + 2x = 30 \implies 5x = 20 \implies x = 4.$$

   So the fifth number is $2x = \boxed{8}$.

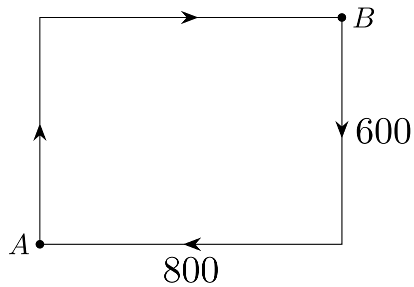
5. How many positive even numbers less than 49 are there whose digits sum to an odd number?

   Answer: 10

   Solution: Similar to the solution for the local version, the $\boxed{10}$ numbers are

$$10, 12, 14, 16, 18, 30, 32, 34, 36, 38.$$

6. Jonathan and Ethan are racing around a rectangular track that is 600 units wide and 800 units long. Jonathan can finish a lap in 4 seconds, while Ethan can finish a lap in 7 seconds. They race one lap around the track, starting at point A and going clockwise. However, once Ethan reaches point B, he cheats by running off of the track, taking the most direct path back to A, at the same speed as before. He still loses the race to Jonathan, who does not cheat. How far away was Ethan from Jonathan when Jonathan finishes the race, *in units*?



   Answer: 800

   Solution: The perimeter of the track is $600 + 800 + 600 + 800 = 2800$ units, so Jonathan runs at $\dfrac{2800}{4} = 700$ units/second and Ethan runs at $\dfrac{2800}{7} = 400$ units/second.

   By the Pythagorean Theorem, the distance along the diagonal is $\sqrt{600^2 + 800^2} = 1000$ units, so the total distance Ethan travels during the race is $600 + 800 + 1000 = 2400$ units, which means he finishes the race in $\frac{2400}{400} = 6$ seconds. Since Jonathan finished the race in 4 seconds, Ethan must have been 2 seconds behind Jonathan, meaning he was a distance of $2 \cdot 400 = 800$ units away on the track.

   Since 800 units is shorter than the length of the diagonal, Ethan is already on the diagonal of the track. So the straight-line distance between Ethan and Jonathan is $\boxed{800}$ units.

7. Isaac picks a number among 1, 2, 3, 4 uniformly at random. Preston picks a number among 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 uniformly at random. What is the probability that Preston picks a strictly larger number than Isaac?

   Answer: $\frac{3}{4}$

   Solution: Note that if Preston chooses a number between 5 and 10, then he automatically has chosen a number higher than Isaac's number. On the other hand,

   - If Preston chooses 1, then Isaac has 0 choices for Preston's number to be strictly larger.
   - If Preston chooses 2, then Isaac's can choose 1. There is 1 choice for Isaac.
   - If Preston chooses 3, then Isaac's can choose $1, 2$. There are 2 choices for Isaac.
   - If Preston chooses 4, then Isaac's can choose $1, 2, 3$. There are 3 choices for Isaac.

So when Preston choose a number between 1 to 4, Isaac has a

$$\frac{0+1+2+3}{4\times 4} = \frac{6}{16}$$

chance of choosing a strictly smaller number.

Since Preston will choose a number uniformly at random, each with possibility $\frac{1}{10}$, there is $\frac{6}{10}$ chance that the Preston's number is between 5 and 10 and $\frac{1}{10}$ chance for Preston to choose each of $1, 2, 3, 4$. Thus, our answer is

$$\frac{6}{10} + \frac{1}{10} \times \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) = \frac{6}{10} + \frac{3}{20} = \boxed{\frac{3}{4}}.$$

8. Helena writes down the number 0 on a chalkboard. Then, every minute afterwards, she counts how many digits in total are on the board and writes down that number. She repeats this until she has written 36 separate numbers on the board (including the first number, 0). For example, if $0, 3, 12$, and 147 were written on the board, there would be four numbers and seven digits. What is the last number Helena writes?

Answer: 60

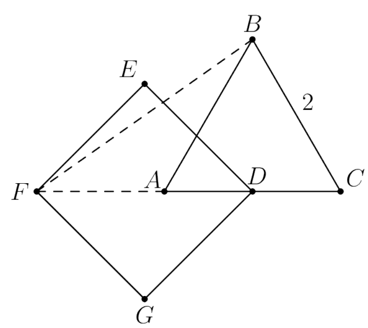
Solution: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are the first 10 numbers.

10 is the eleventh, and then 12, 14, 16, 18, etc.

Helena writes down 25 more two-digit numbers after 10. So the last number is

$$10 + (36 - 11) \cdot 2 = \boxed{60}.$$

9. Let triangle $\triangle ABC$ be an equilateral triangle with side length 2, and let D be the midpoint of side $\overline{AC}$, as shown below. If $DEFG$ is a square with side length $DE = BD$ such that A lies on diagonal $\overline{DF}$, what is the value of BF^2?



Answer: 9

Solution: Since $\triangle ABC$ is an equilateral triangle, $AB = BC = CA = 2$ and $\angle BAC = \angle ABC = \angle BCA = 60°$. If D is the midpoint of side $\overline{AC}$, then $AD = DC = 1$, so

$$\triangle BAD \cong \triangle BCD \qquad \text{(SAS)}.$$

So $\angle BDA = 90°$ and $\angle ABD = \frac{1}{2}\angle ABC = 30°$.

$\triangle ADB$ is a $30-60-90$ right triangle, and the length of $\overline{BD}$ is $\sqrt{2^2 - 1^2} = \sqrt{3}$.

Line segment $\overline{DF}$ is a diagonal of the square, so $\triangle DEF$ is $45-45-90$ right triangle. $\overline{DF}$ has length $\sqrt{2}DE = \sqrt{6}$.

Since $\angle BDF = 90$, the Pythagorean Theorem indicates that

$$BF^2 = BD^2 + DF^2 = 3 + 6 = \boxed{9}.$$

10. Let x and y be positive integers such that $150x^2y$ and $60xy^2$ are both perfect squares and $3xy$ is a perfect cube. Compute the minimum possible value of xy.

Answer: 9000

Solution: The prime factorizations of 150, 60, and 3 are

$$150 = 2 \cdot 3 \cdot 5^2, \qquad 60 = 2^2 \cdot 3 \cdot 5, \qquad 3 = 3^1,$$

respectively. For a number to be a perfect square, its prime factorization must contain primes raised to only even powers. Likewise, for a perfect cube, the powers on every prime factor in its prime factorization must be divisible by 3.

Consider $150x^2y = 2 \cdot 3 \cdot 5^2 x^2 y$, which we know is a perfect square. Since x^2 is already a square number, this means that $y = 2 \cdot 3 \cdot n^2$ where n^2 is a perfect square.

Now consider $60xy^2 = 2^2 \cdot 3 \cdot 5xy^2$, which we know is also a perfect square. Since y^2 is already a square number, this means that $x = 3 \cdot 5 \cdot m^2$ where m^2 is a perfect square.

Finally, consider $3xy = 2 \cdot 3^3 \cdot 5 \cdot n^2 m^2$, which we know is a perfect cube. To make all exponents divisible by 3, n^2m^2 at least equals to $2^2 5^2$. Also, setting $m = 2$, $n = 5$ works.

So the least value of xy is $(3 \cdot 5 \cdot m^2)(2 \cdot 3 \cdot n^2) = \boxed{9000}$.

11. Let $\varphi = \frac{1+\sqrt{5}}{2}$. There exist positive integers a and b such that

$$\sqrt{\varphi} + \sqrt{\frac{1}{\varphi}} = \sqrt{a + \sqrt{b}}.$$

Find $a + b$.

Answer: 7

Solution: First we square both sides of $\sqrt{\varphi} + \sqrt{\frac{1}{\varphi}} = \sqrt{a+\sqrt{b}}$ to get

$$\begin{aligned}
\left(\sqrt{\varphi} + \sqrt{\frac{1}{\varphi}}\right)^2 &= a + \sqrt{b} \\
(\sqrt{\varphi})^2 + 2\sqrt{\varphi}\sqrt{\frac{1}{\varphi}} + \left(\sqrt{\frac{1}{\varphi}}\right)^2 &= a + \sqrt{b} \\
\varphi + 2 + \frac{1}{\varphi} &= a + \sqrt{b}
\end{aligned}$$

We then try to simplify $\varphi + 2 + \dfrac{1}{\varphi}$. First, we would like the denominator of $\dfrac{1}{\varphi}$ to be nicer:

$$\frac{1}{\varphi} = \frac{2}{1+\sqrt{5}} = \frac{2(-1+\sqrt{5})}{(1+\sqrt{5})(-1+\sqrt{5})} = \frac{-2+2\sqrt{5}}{5-1} = \frac{-1+\sqrt{5}}{2}$$

This expression for $\frac{1}{\varphi}$ has a denominator of 2, so we can add it to $\varphi = \frac{1+\sqrt{5}}{2}$:

$$\begin{aligned}\varphi + 2 + \frac{1}{\varphi} &= \frac{1+\sqrt{5}}{2} + 2 + \frac{-1+\sqrt{5}}{2} \\ &= 2 + \frac{1+\sqrt{5}-1+\sqrt{5}}{2} \\ &= 2 + \frac{2\sqrt{5}}{2} \\ &= 2+\sqrt{5}\end{aligned}$$

Therefore $a = 2$, $b = 5$, and $a + b = \boxed{7}$.

12. Aditya chooses a random permutation of the letters that make up "REPOSITORY". What is the probability that Aditya's permutation contains the word "OR" twice? For example, "ORSITYORPE" is one such permutation, but "OROEPSITRY" is not.

Answer: $\mathbf{\frac{1}{45}}$

Solution: To count the permutations that contains the two letters "OR" next to each other, we can consider each "OR" as one block. There are 2 such "blocks", and 6 other letters that aren't O or R, so in total we are arranging $2 + 6 = 8$ items in a row. We have $\binom{8}{2} = 28$ ways to choose two locations out of eight to place the two identical "OR" block.

Of the 6 non-blocked letters $\{E, P, S, I, T, Y\}$, all are distinct, so we have $6! = 720$ ways to permute them. This yields a total of $28 \cdot 720$ permutations with two OR's.

The total number of ways to permute letters in "REPOSITORY" without restrictions is $\frac{10!}{2!^2} = 720 \cdot 1260$ ways, 10! for permuting the 10 letters, and dividing by $2!^2$ because we have two pairs of identical letters, 'O' and 'O', 'R' and 'R'. This gives a probability of

$$\frac{28}{1260} = \boxed{\frac{1}{45}}.$$

13. Call a positive integer n *basic* if there is a positive integer $b > 1$ such that n can be written with b digits in base b (with no leading 0s). For example, 3 is *basic* because it can be written with 2 digits in base 2 as 11_2. How many positive integers $n \leq 2025$ are *basic*?

Answer: 1613

Solution: Let k_b be a number written in base b. We can count how many n are basic in base b for each b, and what the highest basic number is in that base. To have b digits, the number must be at least b^{b-1} and less than b^b. This gives a total of $(b^b - 1) - b^{b-1} + 1 = b^b - b^{b-1}$ for each base b. Note that there is no overlap: we know $b + 1^b > b^b - 1$ always, so a number can only be basic in at most one base. Then, we simply count up until $b^b > 2025$ and add them up:

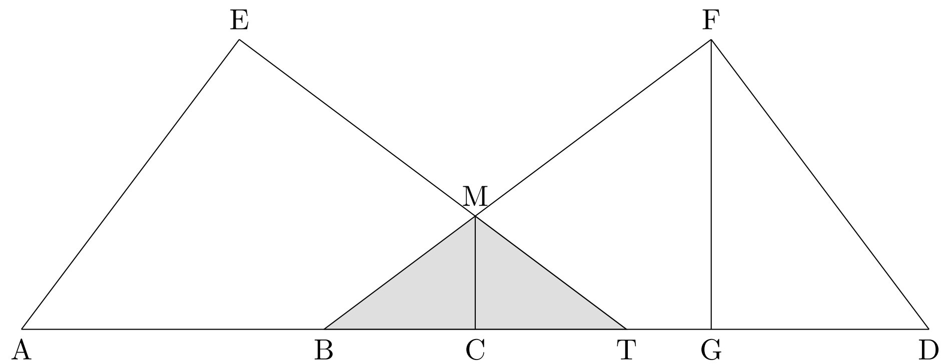
- For $b = 2$, we get $2^2 - 2 = 2$.
- For $b = 3$, we get $3^3 - 3^2 = 27 - 9 = 18$.
- For $b = 4$, we get $4^4 - 4^3 = 256 - 64 = 192$.
- For $b = 5$, we have $5^5 = 3125 > 2025$. Therefore, we stop counting at 2025, and add $2025 - 5^4 + 1 = 2025 - 625 + 1 = 1401$.

The total is then $1401 + 192 + 18 + 2 = \boxed{1613}$.

14. Distinct points A, B, T, and D lie on a line such that $AB = BT = TD = 40$. Points E and F satisfy $AE = DF = 48$ and $TE = BF = 64$, with line segments $\overline{TE}$ and $\overline{BF}$ intersecting at a point M. What is the perimeter of triangle $\triangle BMT$?

Answer: 90

Solution:



Let C and G be the points on the line AD such that MC and AD are perpendicular and FG and AD are perpendicular, as shown in the diagram.

First, notice that, since $\triangle BFD$ is a right-triangle due to the Pythagorean theorem, the area of $\triangle BFD$ is half the product of its two legs, which is $\frac{1}{2}\cdot 64\cdot 48$. Since $BD = BT + TD = 40 + 40 = 80$, this allows us to compute the length of FG, since the area of $\triangle BFD$ can also be written $\frac{1}{2}\cdot 80\cdot FG$. This implies $FG = \frac{192}{5}$.

Since $\triangle BGF$ is a right-triangle, and since $BF = 64$ and $FG = \frac{192}{5}$, by the Pythagorean theorem, $BG^2 + FG^2 = BF^2$, so $BG = \sqrt{64^2 - (\frac{192}{5})^2} = \frac{256}{5}$.

Since $\triangle BCM$ and $\triangle BGF$ are similar triangles, $FG/MC = BG/BC$. Notice also $BC = \frac{1}{2}BT = 20$, so we can use this equation to compute MC. We find $MC = 15$, $BM = MT = 25$

Since $MC = 15$ and $BT = 40$, we conclude that the area of $\triangle BMT$ is $\frac{1}{2}\cdot 15\cdot 40 = 300$.

The perimeter is $BM + MT + BT = 25 + 25 + 40 = \boxed{90}$.

15. For integers a and b, define the binary operation $\bigstar$ by

$$a \bigstar b = a + b + ab.$$

There exists an associative operation $\blacktriangledown$, meaning that $(a \blacktriangledown b) \blacktriangledown c = a \blacktriangledown (b \blacktriangledown c)$ for all real numbers a, b and c, such that whenever x is a non-negative integer,

$$x \bigstar y = \underbrace{y \blacktriangledown \cdots \blacktriangledown y \blacktriangledown y}_{x\ \blacktriangledown\text{'s}}.$$

Compute $6 \blacktriangledown 7$.

Answer: 14

Solution 1: Notice that $x \bigstar 0 = x + 0 + (x \times 0)$, which is equal to x. So $x = x \bigstar 0 = \underbrace{0\blacktriangledown \cdots \blacktriangledown 0 \blacktriangledown 0}_{\text{x+1 times}}$.

So $6\blacktriangledown 7 = (\underbrace{0\blacktriangledown \cdots \blacktriangledown 0 \blacktriangledown 0}_{\text{7 times}})\blacktriangledown(\underbrace{0\blacktriangledown \cdots \blacktriangledown 0 \blacktriangledown 0}_{\text{8 times}})$, which by associativity is equal to $\underbrace{0\blacktriangledown \cdots \blacktriangledown 0 \blacktriangledown 0}_{\text{15 times}}$, which is equal to $14\bigstar 0$, which is $\boxed{14}$.

Solution 2:

Notice that $\bigstar$ is multiplication with the indexes shifted up by one:

$$a\bigstar b = (a+1)(b+1) - 1.$$

We know regular multiplication is defined by repeated addition. Since $\bigstar$ is also repeated $\blacktriangledown$, we can guess that $\blacktriangledown$ has to do with addition.

Try addition shifted up by one and see if it works: Let us assume

$$a\blacktriangledown b = (a+1) + (b+1) - 1,$$

and check whether the required equation still holds:

$$x\bigstar y \stackrel{?}{=} \underbrace{y\blacktriangledown \ldots \blacktriangledown y \blacktriangledown y}_{x+1 \text{ times}}.$$

We can start with $y\blacktriangledown y$, $y\blacktriangledown y\blacktriangledown y$ and see if we can find a pattern:

$$y\blacktriangledown y = (y+1) + (y+1) - 1 = 2y + 1,$$

$$y\blacktriangledown y\blacktriangledown y = (2y+1+1) + (y+1) - 1 = 3y + 2,$$

$$\ldots$$

In general, it looks like if there are $(x+1)$ copies of y with $\blacktriangledown$ in between, we get

$$\underbrace{y\blacktriangledown \ldots \blacktriangledown y \blacktriangledown y}_{x+1 \text{ times}} = (x+1)y + x.$$

This multiplies out to

$$\underbrace{y\blacktriangledown \ldots \blacktriangledown y \blacktriangledown y}_{x+1 \text{ times}} = (x+1)y + x = x + y + xy = x\bigstar y.$$

So our guess was correct. Therefore, $6\blacktriangledown 7 = (6+1) + (7+1) - 1 = \boxed{14}$.

16. Square $ABCD$ has side length $4 + 2\sqrt{2}$. A circle is drawn such that it passes through C and is tangent to sides $\overline{AB}$ and $\overline{AD}$. The area of the total region covered by at least one of the circle and the square can be written as $a + b\sqrt{c} + d\pi$, where a, b, c, and d are all integers, and c is square-free (it is not divisible by any perfect square greater than 1). Find $a + b + c + d$.

Answer: 34

Solution: Similar to the solution for the local version, the area is

area of (square $ABCD -$semicircle $GCH+$semicircle $GCH) = (4+2\sqrt{2})^2 - 16 + 8\pi = 8 + 16\sqrt{2} + 8\pi$.

The total is $8 + 16 + 2 + 8 = \boxed{34}$.

17. What is the least positive number of zeroes that can be concatenated to the end of 2025 such that the sum of the even factors of the resulting number is divisible by 45?

    Here "concatenated" means writing zeroes at the end of the number.

    For example: Concatenating one zero to "2025" gives "20250", concatenating two zeroes gives "202500", and so on.

    Answer: 12

    Solution: Assuming we have k digits after 2025, the prime factorization of our number is

$$2025 \cdot 10^k = 2^k 3^4 5^{k+2}.$$

To find the sum of the even factors of this number we use the following equation coupled with the formula for a geometric series:

$$\begin{aligned}(2 + 2^2 + \cdots + 2^k)&(1 + 3 + \cdots + 3^4)(1 + 5 + \cdots + 5^{k+2})\\ &= (2^{k+1} - 2)(121)\left(\frac{5^{k+3} - 1}{4}\right)\\ &= (2^k - 1)(121)\left(\frac{5^{k+3} - 1}{2}\right)\end{aligned}$$

The factor 121 is coprime with 45, so we can disregard this factor in the following discussion of which k value makes the remaining part divisible by 45.

In order for this value to be divisible by 45, at least one of its factors must be divisible by 5. Since 5^{k+3} is divisible by 5, $5^{k+3} - 1$ is not divisible by 5, so $\frac{5^{k+3}-1}{2}$ is also not. Then the other factor $2^k - 1$ must be divisible by 5.

$$2^1 - 1 = 1, \qquad 2^2 - 1 = 3, \qquad 2^3 - 1 = 7, \qquad 2^4 - 1 = 15 \equiv 0 \pmod 5.$$

If we do long division by 4 with quotient q and remainder r, we can write the exponent k uniquely as $k = 4q + r$, where $r = 0, 1, 2, 3$. Then

$$2^k = 2^{4q+r} \equiv \left(2^4\right)^q 2^r \equiv 1 \cdot 2^r \equiv 2^r \pmod 5.$$

Among $r = 0, 1, 2, 3$, the only choice for r such that $2^k \equiv 2^r \equiv 1 \pmod 5$ is $r = 0$, so $k = 4q$ for some integer q. In other words, $2^k - 1 \equiv 0 \pmod 5$ if and only if k is divisible by 4.

The other condition for the sum of all even factors to be divisible by 45 is that it is also divisible by 9. Since k must be divisible by 4, $k + 3$ is odd.

For an arbitrary odd number in the form $2x + 1$ or an arbitrary even number in the form $2x$,

$$5^{2x} = \left(5^2\right)^x \equiv 1^x \equiv 1 \pmod 3, \qquad 5^{2x+1} = 5 \cdot \left(5^2\right)^x \equiv 2 \cdot 1 \equiv 2 \pmod 3.$$

Therefore, even powers of 5 are $1 \pmod 3$ and odd powers of 5 are not $1 \pmod 3$. Since $k + 3$ is odd, $5^{k+3} - 1$ is never divisible by 3, so $5^{k+3} - 1$ is never divisible by 9. Thus, $2^k - 1$ must be divisible by 9. We check that powers of 2 mod 9 are

$$2^0 \equiv 1, \quad 2^1 \equiv 2, \quad 2^2 \equiv 4, \quad 2^3 \equiv 8, \quad 2^4 \equiv 7, \quad 2^5 \equiv 5 \pmod 9,$$

and if we can write a power of 2 as 2^{6q+r}, $r = 0, 1, 2, 3, 4, 5$, we get

$$2^{6q+r} \equiv \left(2^6\right)^q 2^r \equiv 1^q \cdot 2^r \equiv 2^r \pmod 9$$

so $2^{6q+r} \equiv 2^r \equiv 1 \pmod 9$ iff $r = 0$. This is the same as saying the exponent of 2 is in the form $6q$, i.e. $2^k - 1$ is divisible by 9 if and only if k is divisible by 6.

Thus, for the sum of all even factors to be divisible by 45, k must be a multiple of both 4 and 6. The least common multiple is 12, so the final answer is $\boxed{12}$.

18. What is the greatest possible value of C satisfying the property that the following system of equations

$$x^2 = y + C, \quad y^2 = x + C$$

has exactly 2 real solutions, and all solutions are real?

Answer: $\mathbf{\frac{3}{4}}$

Solution 1: By substituting $y = x^2 - C$ into $y^2 = x + C$, we get the fourth-degree polynomial

$$\begin{aligned}(x^2 - C)^2 &= x + C \\ x^4 - 2Cx^2 - x + (C^2 - C) &= 0\end{aligned}$$

Note that if x is real, then y is real as well. Since we are given that all solutions are real, we only need to consider the case where x is real. Let r_1, r_2 denote the two unique solutions to the polynomial above. When factored, the $x^4 - 2Cx^2 - x + (C^2 - C)$ is either

$$(x - r_1)^1(x - r_2)^3, \quad (x - r_1)^2(x - r_2)^2, \quad \text{or } (x - r_1)^3(x - r_2)^1.$$

$(x - r_1)^1(x - r_2)^3$ can be ignored since it is equivalent to $(x - r_1)^3(x - r_2)^1$ by swapping the values of r_1 and r_2.

- Suppose the $x^4 - 2Cx^2 - x + (C^2 - C)$ can be written in the form $(x - r_1)^2(x - r_2)^2$. Since the coefficient of x^3 is 0, this implies that

  $$\begin{aligned}-(2r_1 + 2r_2) &= 0 \\ r_2 &= -r_1\end{aligned}$$

  Similarly, since the coefficient of x is -1, this implies that

  $$-(2r_1^2 r_2 + 2r_1 r_2^2) = -1$$

  Pluggin in $r_2 = -r_1$,

  $$-(2r_1^2 r_2 + 2r_1 r_2^2) = 2r_1^3 - 2r_1^3 = 0 \neq -1$$

  so this equation we set up from the coefficient of x cannot be satisfied. Thus, $x^4 - 2Cx^2 - x + (C^2 - C)$ cannot be factored in the form $(x - r_1)^2(x - r_2)^2$.
- The other possible factored form of $x^4 - 2Cx^2 - x + (C^2 - C)$ (up to ordering) is $(x - r_1)^3(x - r_2)^1$. Based on the coefficients of x^3:

  $$-(3r_1 + r_2) = 0 \Rightarrow r_2 = -3r_1$$

Based on the coefficient of x:

$$-(3r_1^2 r_2 + r_1^3) = -1$$
$$-8r_1^3 = 1$$
$$r_1 = -\frac{1}{2}$$

Thus, $r_2 = \frac{3}{2}$, so the polynomials is

$$\left(x + \frac{1}{2}\right)^3 \left(x - \frac{3}{2}\right) = x^4 - \frac{3}{2}x^2 - x - \frac{3}{16}$$

Since $x^4 - 2Cx^2 - x + (C^2 - C)$ matches the polynomial above for $C = \frac{3}{4}$, $C = \frac{3}{4}$ is the only value of C that meets the criteria specified above.

Thus, the final answer is $C = \boxed{\frac{3}{4}}$.

Solution 2:

If we substitute in y for x, we get a degree-4 polynomial, suggesting that this system of equations has 4 solutions, real or non-real.

These solutions can actually be obtained through some substitutions!

- The most obvious substitution is $y = x$, which turns both equations into $x^2 = x + C$. Using the quadratic formula gives $x = y = \frac{1 \pm \sqrt{1+4C}}{2}$. These roots are distinct and real when $C \geq -\frac{1}{4}$, identical and real when $C = -\frac{1}{4}$, and distinct and non-real when $C \leq -\frac{1}{4}$.
- The other substitution is $y = -1 - x$, which turns both equations into $x^2 = (-1 - x) + C$. Using the quadratic formula gives $x = y = \frac{-1 \pm \sqrt{4C-3}}{2}$. These roots are d istinct and real when $C \geq \frac{3}{4}$, identical and real when $C = \frac{3}{4}$, and distinct and non-real when $C \leq \frac{3}{4}$.

For all $C > \frac{3}{4}$, the system of equations has 4 distinct real solutions. When $C = \frac{3}{4}$, the first substitution yields the solutions $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$, and the second substitution yields the solution $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, which is identical to another of the solutions. Therefore, when $C = \frac{3}{4}$, the system of equations has exactly 2 real solutions, so the greatest possible value of C is $\boxed{\frac{3}{4}}$.

Solution 3: This solution appeals to geometric intuitions but is not rigorous.

Each of the two equations

$$x^2 = y + C, \quad y^2 = x + C$$

is a parabolas in the $xy-$plane. We are asking for what values of C would these parabolas intersect at exactly two points.

Try a few values of C; at $C = 0$ they intersect at two points, and we can see as C decreases the parabola will become disjoint. Try $C = 1$; the parabolas will intersect at four points. As C increase from 1, we will always have four points of intersections.

Note: When you have four points of intersection, one pair of them are on the $y = x$ line, the other pair is on the $x + y = -1$ line. This also hints at the substitutions we did in Solution 2.

When the two parabola has two intersections, at the intersection in the third quadrant, the slope of $x^2 = y + C$ is less steep than the slope of $y^2 = x + C$, and vice versa when they have four intersection. At the value between 0 and 1 where any larger C will give four intersections, the parabolas should have the same slope. The two parabolas are also mirror images of each other across the $y = x$ line, so at the point of intersection, both should be going in the direction as the $y = -x$ line, forming a 90 degree angle with the $y = x$ line.

For which x value on the parabola $x^2 = y + C$ would it be going in the direction of $y = -x$? The parabola $x^2 = y + C$ is a vertical shift of $y = x^2$, so we consider the parabola $y = x^2$ instead. Use $y = -x + k$ to denote a generic line in the direction of $y = -x$. We want to solve for the value k such that $y = x^2$ and $y = -x + k$ has only one point of intersection:

$$(-x + k) = x^2 \implies x^2 + x - k = 0$$

$$x^2 + x - k = 0 \text{ has only one solution } \implies b^2 - 4ac = 1 + 4k = 0;$$

$$x = \frac{-b}{2a} = -\frac{1}{2}$$

Since $x^2 = y + C$ and $y^2 = x + C$ intersect on the line $y = x$, this point of intersection has coordinate $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. The value of C such that the parabolas goes through $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ is $C = \boxed{\frac{3}{4}}$.

19. Consider a bee (denoted by X) in a rectangular honeycomb as seen below:

	1	2	3	4	5	6	7
A				X			
B							
C							
D							

In one move, the bee may move to an adjacent square via an up, down, left, or right move, and it can no longer move once it reaches row D. The bee cannot move outside the honeycomb. It cannot revisit a square it has already been to, and it cannot move more than six times. Find the number of different paths the bee can take from its starting point to row D.

Answer: 63

Solution 1: The bee must move down exactly 3 times, so it may make a horizontal move at most $6 - 3 = 3$ times. Casework can be done in the number of horizontal moves the bee makes.

- 0 horizontal moves - The bee can only move down, so there is only 1 path.
- 1 horizontal move - The bee could move left or right on rows A, B, or C, for a total of $2 \cdot 3 = 6$ total paths.
- 2 horizontal moves - the bee does exactly one of the following:
  - Two horizontal moves on the same row: Since the bee cannot go to a square it has already been at, it must make two moves in the same direction. Similar to the case with one horizontal move, there are 3 rows where the bee can make its horizontal moves and 2 directions the bee can move in, for a total of 6 paths.

  - One horizontal moves each on two different rows: There are 3 ways to pick the 2 rows that the bee makes a horizontal move on. On each row the bee can move left or right, so $2 \cdot 2 = 4$ ways for the bee to make horizontal moves once it has chosen which rows to make horizontal moves on, for a total of 12 paths.

  In total, there are $6 + 12 = 18$ paths with exactly 2 horizontal moves.

- 3 horizontal moves - the bee does exactly one of the following:
  - 3 horizontal moves on one row: similar to the previous cases, there are 3 different rows that can be chosen and 2 horizontal directions the bee can move in, for a total of 6 paths.
  - 2 horizontal moves on one row and 1 horizontal move on the other: There are $3 \cdot 2$ ways to choose the rows (the rows are distinguished now since one has 2 horizontal moves and the other only has 1), and $2 \cdot 2 = 4$ ways to choose horizontal moves on these rows, for a total of 24 paths.
  - 1 horizontal move in each of A, B, and C. There are 2^3 ways to choose the horizontal moves, and only 1 set of rows, for a total of 8 paths.

  In total, there are $6 + 24 + 8 = 38$ paths with 3 horizontal moves.

Since the cases above are mutually exclusive, the total number of paths is $1 + 6 + 18 + 38 = \boxed{63}$ paths.

Solution 2:

Notice that the only way to reach a cell in row D is from the cell directly above it in row C. Hence, we can instead count the number of ways to reach any cell in row C in 5 or fewer moves without the restriction that we can't move left or right in the target row. The number of ways to reach row C without this restriction is equal to the number of ways to reach row D with our original rules.

Then, notice that because the grid is symmetrical, every path to C1, C2, or C3 can be mirrored to get paths to C5, C6, or C7, so we can just calculate the number of paths to C1, C2, and C3 and double it to get the total for all 6 cells.

Finally, notice that it takes exactly 5 moves to reach C1, 4 moves to reach C2, 3 or 5 moves to reach C3, and 2 or 4 moves to reach C4. This reduces the number of paths we need to check.

Now we will count the number of ways to get to each cell in row C in 5 or fewer moves and sum them all up to get the total number of paths to row C.

- C1 can only be reached in exactly 5 moves, and exactly 2 of them must be moving down, so the number of ways to reach C1 is $\binom{5}{2} = \frac{5!}{2!3!} = \mathbf{10}$.
- C2 can only be reached in exactly 4 moves, and exactly 2 of them must be moving down, so the number of ways to reach C2 is $\binom{4}{2} = \frac{4!}{2!2!} = \mathbf{6}$.
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  - For a path to reach C3 in exactly 3 moves, it must move down exactly 2 times and left exactly once. Thus, the number of ways to reach it is $\binom{3}{2} = \frac{3!}{2!} = 3$.
  - For a path to reach C3 in exactly 5 moves, we can partition it by counting the number ways of reaching C2, C4, and B3 in 4 moves without moving through C3.
    * The only way to reach C2 in 4 moves without moving through C3 is to move through B2. The number of ways to reach B2 is $\binom{3}{2} = \frac{3!}{2!} = 3$.
    * The only way to reach B3 in 4 moves is to either move left, left, down, right, or right, down, left, left. Thus, there are 2 ways to reach B3 in 4 moves.

    * For C4, we can again partition by counting the number of ways to reach C5 and B4 in 3 moves without touching C4.

    Doing this, we find 2 ways to reach C5 and 2 ways to reach B4, for a total of 4 ways to reach C4. Thus, are 9 ways to reach C3 in exactly 5 moves.

  This gives us a total of **12** ways to reach C3.

- – There is only 1 way to reach C4 in exactly 2 moves: moving down twice.
  – To count the number of ways to reach C4 in exactly 4 moves, we can partition again and use calculations from above: the number of ways to reach B4 in 3 moves is 2, and the number of ways to reach C5 without touching C4 is 2. We can mirror the paths to C5 to get the paths to C3, giving us a total of 6 paths to C4 in exactly 4 moves.

  Hence, there are **7** ways to reach C4.

Summing everything together and doubling the paths we got from C1, C2, and C3, the total number of ways to move from cell A4 to row C is $2(10+6+12)+7=63$. Since the number of ways to reach row C in 5 or fewer moves is equal to the number of moves to reach row D in 6 or fewer moves, as we established before, the total of ways to move from cell A4 to row D in 6 or fewer moves is $\boxed{63}$.

20. Let ω_1 be a circle with center O and radius 4 and ω_2 be a circle with center P and radius 1 such that ω_1 and ω_2 are externally tangent to each other and both tangent to line $\overleftrightarrow{AB}$, with ω_1 tangent at A, and ω_2 tangent at B. Point D lies on ω_2 such that O, P, and D are collinear and D is not on ω_1. Line ℓ is tangent to circle ω_2 at D. Let C be the point of intersection of line $\overleftrightarrow{OA}$ and line ℓ, and let K be the point of intersection of line $\overleftrightarrow{AB}$ and line ℓ. If Q is a point inside triangle $\triangle AKC$ such that triangle $\triangle OQD$ is larger than $\triangle AQC$, what is the area of the region of possible locations of Q?

Answer: 11

Solution: Similar to the solution for the local version, all other calculations goes the same. Because $OD = AC = 6$, we want the region of Q such that Q is closer to line AC than line OD. It is the area of $AMNC$ such that ON is an angle bisector of $\angle COD$.

$$AM = 2, MK = AK - AM = \frac{3}{4}AC - AM = 4.5 - 2 = 2.5$$

By length chasing, we can find that if ON is an angle bisector of $\angle COD$, then

$$OH = OD = 6, HN = ND = 3, HC = 4, NC = 5$$

So $AH = 2$. This is the height of $\triangle MNK$ with base MK.

The area of triangle $\triangle MNK$ is

$$\triangle MNK = \frac{1}{2} \cdot MK \cdot AH = \frac{1}{2} \cdot 2.5 \cdot 2 = \frac{5}{2}.$$

The total area of $\triangle AKC$ is $\frac{27}{2}$, so the area of $AMNC = \frac{27}{2} - \frac{5}{2} = \boxed{11}$.