1. What is the largest number Jessica can create using each of $1, 2$, and 6 at most once along with the operations $+$, $-$, $\times$, $\div$, and parentheses? Note that combining digits (e.g., making 16 from 1 and 6) is not allowed.

   Answer: 18

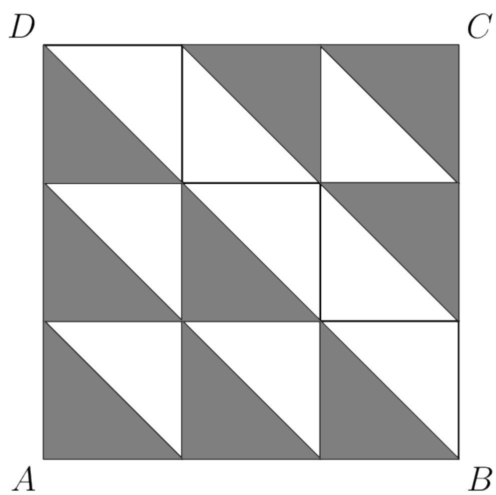
   Solution: Subtracting and dividing usually shrinks our answer, so we want to be using + and $\times$. Furthermore, multiplying by 1 does nothing, so we should be adding by 1.

   The only possibilities are $1+2+6 = 9$, $1+(2\times 6) = 13$, $(1+2)\times 6 = 18$, and $2\times(1+6) = 14$.

   So largest number we can get will be $(1+2)\times 6 = \boxed{18}$.

2. Let N_1 be the answer to Problem 1.

   If the area of square $ABCD$ below is N_1, find the area of the shaded region.



   Answer: 9

   Solution 1: The most satisfying way solve this problem is to notice the following: in each of the 9 squares, one of the two triangles is shaded. Therefore, the area of the shaded region is half the area of the square, which is $\frac{1}{2}\cdot 18 = \boxed{9}$.

   Solution 2: Alternatively, one can count 9 shaded triangles, 6 in the bottom left and 3 in the top right. Each has area $\frac{1}{2}$ of a square and there are 9 squares, so the area is half the area of the square or also $\frac{1}{2}\cdot 18 = \boxed{9}$.

3. An ancient numbering system has three symbols: smiles [$\smile$], frowns [$\frown$], and dots [$\cdot$]. Each number is represented by a group of dots on top of any amount of frowns or smiles. Each dot counts as 1, each frown counts as 5, and each smile counts as 20. For example, $\overset{\cdot}{\frown}$ represents 6. What is the numerical value of the following expression?

$$\overset{\cdot\cdot}{\underset{\smile}{\smile}} \times (\overset{\cdot\cdot}{\frown} \times \overset{\cdot\cdot}{\frown} - \cdot) + (\overset{\cdot\cdot}{\smile} - \overset{\cdot\cdot}{\frown} - \overset{\cdot\cdot}{\frown} + \cdot)$$

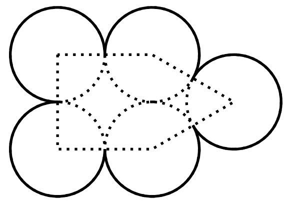
   Answer: 2025

   Solution 1: Compute the expression to get $42\times(7\times 7-1)+(22-7-7+1) = 42\times 48+9 = 2016+9 = \boxed{2025}$.

   Solution 2: Compute the expression and notice that by difference of squares, $42\times 48+9 = 45^2-3^2+9 = 45^2 = \boxed{2025}$.

4. Let N_3 be the **square root** of the answer to Problem 3.

   Consider a convex pentagon with two adjacent right angles and all sides having integer length N_3, as shown in the diagram below. Evan draws a circle centered at each vertex such that each circle is tangent to the circles centered at adjacent vertices and all circles are congruent. If the perimeter of the resulting shape, shown in bold, is $\frac{a\pi}{2}$, find a.



   Answer: 315

   Solution: Each side of the pentagon has length $N_3 = \sqrt{2025} = 45$. Since each side is the combined radii of 2 circles, the radius of each circle is $\frac{45}{2}$.

   Our total perimeter is the circumference of all 5 circles minus the arcs that are inside of the pentagon. Each circle has a circumference of $2\pi r = 2\pi\left(\frac{45}{2}\right) = 45\pi$, totaling $45\pi \cdot 5 = 225\pi$.

   Each arc inside of the pentagon corresponds to an interior angle of the pentagon. Since the sum of the interior angles of a pentagon is $(5-2)\cdot 180° = 540°$, the total length of the arcs inside the circle are $\frac{540°}{360°} \cdot 45\pi = \frac{3}{2} \cdot 45\pi = \frac{135\pi}{2}$.

   Our perimeter is $225\pi - \frac{135\pi}{2} = \frac{450\pi}{2} - \frac{135\pi}{2} = \frac{315\pi}{2}$, so our answer is $\boxed{315}$.

5. Let N_4 be the **rightmost digit** of the answer to Problem 4.

   How many positive factors of $\frac{2025}{N_4}$ have no digits that are prime numbers?

   Answer: 3

   Solution: With $N_4 = 5$, we need the factors of $\frac{2025}{5} = 3^4 \cdot 5$. Because all of the factors of 405 are odd, any factor that is a multiple of 5 will end in a 5. But since 5 is prime, we only need to consider the other factors. So, check the powers of 3 up to 3^4, which are $1, 3, 9, 27, 81$. Only 1, 9, and 81 have no prime digits, so our answer is $\boxed{3}$.

6. Aarush writes down the following equations involving three positive numbers A, B, and K:

$$K \cdot A \cdot B \cdot A \cdot B = \frac{4}{9},$$
$$A \cdot B \cdot A \cdot K \cdot A \cdot B \cdot A = \frac{1}{9},$$
$$B \cdot A \cdot A \cdot B \cdot A \cdot A = \frac{1}{36},$$

   where the left hand side of each equation is a product of A's, B's, and K's. Determine the value of Aarush's constant, defined as $B \cdot A \cdot B \cdot B \cdot 1 \cdot 3 \cdot 5$.

   Answer: $\frac{20}{9}$

   Solution: If we divide the second equation by the first equation, we see that $A^2 = \frac{1}{4}$. A must be positive, so $A = \frac{1}{2}$. Substituting this into the third equation, we get $B^2 = \frac{16}{36}$ which gives $B = \frac{2}{3}$. Since we don't need to know the value of K to find our answer, we can calculate $B \cdot A \cdot B \cdot B \cdot 1 \cdot 3 \cdot 5 = \boxed{\frac{20}{9}}$.

7. Let N_6 be the **denominator** when the answer to Problem 6 is written in simplest form. If your answer to Problem 6 is a mixed number, convert it to an improper fraction to find N_6.

   Oliver is planning a "world tour" in which he plans on visiting N_6 places, including his starting position, on the surface of the Earth, selected uniformly at random. Assuming Earth is a perfect sphere and that the equator divides it into two congruent hemispheres, what is the probability that Oliver must cross the equator at some point during his tour?

   Answer: $\mathbf{\frac{255}{256}}$

   Solution: This can easily be generalized to see that for any N points placed on the sphere, the chance of the perimeter of the polygon created by the points intersecting the equator is $1-2^{1-n}$. We can see that for every additional point added, there is a $\frac{1}{2}$ chance it is on the same hemisphere as our initial point. Since we are looking for the probability that he does not have to cross the equator, it is 1-(this aforementioned probability).

   Thus, using $N_6 = 9$, we have $1 - 2^{(1-9)} = 1 - \frac{1}{256} = \boxed{\frac{255}{256}}$.

8. Let N_7 be the answer to Problem 7.

   How many circles are tangent to all three lines $y = \frac{x}{N_7}$, $y = N_7(x+1)$, and $y = N_7 - x$?

   Answer: 4

   Solution: We can see by a simple graph that there are only 4 places where a circle can touch all three lines. With our given case of N_7, the three lines create a triangle, who's incircle gives us our first solution, then we can see three more solutions with circles tangent to each leg of the triangle externally. This gives us $\boxed{4}$ solutions.

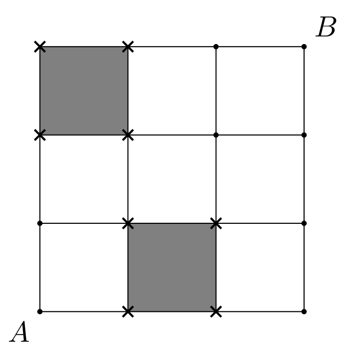
9. Let N_6 be the answer to Problem 6.

   Define the *naive sum*, denoted by $\oplus$, of any two simplified fractions to be $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$, where a, b, c, and d must be positive integers. For example, $\frac{1}{3} \oplus \frac{3}{5} = \frac{4}{8} = \frac{1}{2}$. Let $\frac{p}{q}$ be the smallest simplified fraction such that $N_6 \oplus \frac{p}{q} \geq N_6$. Find $\frac{p}{q}$.

   Answer: $\mathbf{\frac{20}{9}}$

   Solution: We have $\frac{20+p}{9+q} \geq \frac{20}{9} \implies \frac{p}{q} \geq \frac{20}{9}$. Thus $\frac{p}{q} = \boxed{\frac{20}{9}}$.

10. Arthur the Ant lives at point A on a 4×4 grid of points and can move only to adjacent points connected by line segments. However, Shrey the Snail blocks two of the nine squares, making all vertices of those squares inaccessible to Arthur. If the order that Shrey blocked them does not matter, how many ways can Shrey choose two distinct squares to block such that Arthur can still travel from A to B?



One example way that leaves no possible path from A to B.

Answer: 12

Solution: First notice that if Shrey blocks the squares containing A or B, no path is possible. Next, notice that if the two blocked squares are on opposite sides of the diagonal AB, no path is possible. Thus, to have a possible path, we must consider the case when both blocked squares are on the same side of AB and not in the bottom-left or top-right corner.

If we consider the bottom-right corner, there are 3 squares available to place blocks on, and we know Shrey must block 2 of them. Because order of blocking does not matter, there are $\binom{3}{2} = 6$ ways to place the blocks in the bottom-right corner. By symmetry, the top-left case is identical. Thus, the answer is $6 + 6 = \boxed{12}$.

11. Let N_{10} be the answer to Problem 10.

Rectangle $ABCD$ has $AB = 18$ and $BC = 24$. Harsh draws point X on $\overline{AD}$ such that $AX = N_{10}$. Let Y and Z be points on $\overline{BD}$ such that $BY = YZ = ZD$. Find the area of triangle $\triangle XYZ$.

Answer: 36

Solution: The area of triangle ABD is $\frac{1}{2}(18)(24) = 216$. The area of triangle AYZ is $\frac{1}{3}[ABD] = 72$.

Focus on similar right triangles $\triangle AA'D \sim \triangle XX'D$, where A' and X' are formed by altitudes from A and X to base $\overline{BD}$. Lengths AA' and XX' are in a $\frac{24}{24-N_{10}}$ ratio, which gives us that the area of triangle XYZ is $\frac{24-N_{10}}{24}[AYZ] = 72 - 3N_{10}$.

Using $N_{10} = 12$, we get our answer $72 - 36 = \boxed{36}$.

12. Let N_{10} be the answer to Problem 10 and N_{11} be the answer to Problem 11.

Call a pentagon $ABCDE$ *tri-right* if all of its interior angles are less than $180°$, all of its side lengths are positive integers, $AB = CD$, and $\angle B = \angle C = \angle E = 90°$. Reflections and rotations of a pentagon are not considered distinct. What is the number of distinct *tri-right* pentagons with perimeter at most $4N_{10}$ such that side length AB is a multiple of $\sqrt{N_{11}}$?

Answer: 7

Solution: We are considering perimeters at most 48. Additionally, since $AB = CD \geq 6$, the perimeter of the remaining sides is at most $48 - 6 - 6 = 36$.

Notice that each pentagon consists of a right triangle ADE and a rectangle $ABCD$. So, the perimeter is just $AB + BC + CD + DE + AE = [ADE] + AB \times 2$ where $[ADE]$ is the perimeter of ADE.

So, lets count the pythagorean triples with perimeter less than or equal to 36, there are not that many.

$3 - 4 - 5$ has perimeter 12, so $AB = 6, 12, 18$ is 3 possibilities.

$6 - 8 - 10$ has perimeter 24 so $AB = 6, 12$ is 2 possibilities.

$9 - 12 - 15$ has perimeter 36 so $AB = 6$ is 1 possibility.

$12 - 16 - 20$ is too large.

$5 - 12 - 13$ has perimeter 30 so $AB = 6$ has 1 possibility.

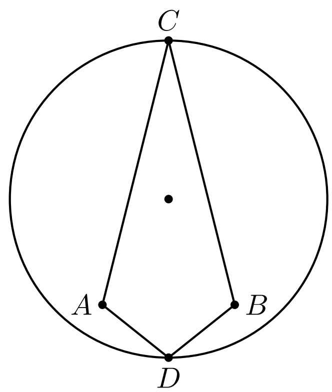
$10 - 24 - 26$ is too large.

$8-15-17$ and $5-24-25$ are too large.

Thus, our answer is $3+2+1+1=\boxed{7}$.

13. Let N_{14} be the answer to Problem 14.

Let O be a circle with radius N_{14} as shown below, and let $\overline{CD}$ be a diameter of circle O. Let A and B be points inside the circle such that $\overline{AB} \perp \overline{CD}$. If the ratio of the area of circle O to the area of quadrilateral $ACBD$ is $\frac{6}{5}\pi$, what is AB?



Answer: 50

Solution: Let the answer to this problem be N_{13}. Since $\overline{CD}$ is the diameter of ω, the combined heights of the two triangles add up to $2N_{14}$, resulting in a total area of $2 \cdot \frac{N_{13}N_{14}}{2} = N_{13}N_{14}$. And the area of the circle is πN_{14}^2, we can set up the equation

$$\frac{6}{5}\pi \cdot N_{13}N_{14} = \pi N_{14}^2$$
$$N_{14} = \frac{6}{5}N_{13} \text{ (since } N_{14} \text{ is non-zero)}$$

And given what we derived from Problem 14 that

$$N_{13} = \frac{140+N_{14}}{4},$$

we can solve for N_{13} to be $\boxed{50}$.

14. Let N_{13} be the answer to Problem 13.

Mary, Sabine, Meghan, and Justin each take a quiz. The sum of Mary's score, Sabine's score, and Meghan's score is 140. The average of all four scores is N_{13}. What score did Justin get?

Answer: 60

Solution: Let the answer to this problem be N_{14}. We have

$$N_{13} = \frac{140+N_{14}}{4},$$

and since we have

$$N_{14} = \frac{6}{5}N_{13}$$

from Problem 13, we solve for the set of equations and get $N_{14} = \boxed{60}$.

15. A team of 60 gardeners is planting trees. The 60 gardeners are split into groups A, B, and C with a, b, and c members, respectively. If, every day, each member in group A plants a trees, each in B plants b trees, and each in C plants c trees, then they can plant 10320 trees in 5 days. If their planting rates are swapped so that each person in group A plants b trees, each in B plants c, and each in C plants a, how many trees can the 60 gardeners plant in one day?

    Answer: 768

    Solution: This is a work and algebra problem. We know that $a+b+c=60$ from the volunteer team size. Under the first set of work rates, the team can plant $a^2+b^2+c^2$ trees per day. Since this takes them 5 days at this rate to plant 10320 trees, we know that $a^2+b^2+c^2=10320/5=2064$.

    Under the second set of work rates, the team can plant $ab+bc+ac$ trees per day. Since $(a+b+c)^2-(a^2+b^2+c^2)=2(ab+bc+ac)$, our answer is $\frac{1}{2}(60^2-2064)=\frac{1}{2}(1536)=\boxed{768}$ trees per day.

16. Let N_{15} be the **largest prime factor** of the answer to Problem 15.

    Viraj lists out all the fractions $\frac{a}{b}$ such that the sum of positive integers a and b is less than or equal to N_{15}^2. He then erases all the expressions that are not in simplest form. Note that for this problem, $\frac{5}{1}$ and $\frac{2}{4}$ are **not** in simplest form, but $\frac{7}{3}$ is in simplest form. If the product of all the remaining fractions is $\frac{1}{P}$, determine the largest perfect cube that is a factor of P.

    Answer: 64

    Solution: Notice that if $\frac{a}{b}$ is in simplest form, so is $\frac{b}{a}$ unless $a=1$. So, multiplying them all will result in 1, with the exception of fractions like $\frac{1}{n}$. Since $768=2^8\cdot 3$, we use $N_{15}=3$. So, take the product of all $\frac{1}{n}$ where $n=2,\dots,3^2-1$. So $P=8!=2^7\cdot 3^2\cdot 5\cdot 7$. The largest cube factor of P is thus $2^6=4^3=\boxed{64}$.

17. The number 2025 can be expressed as a sum using only prime numbers between 100 and 110. What is the largest number of times that 109 can appear in such a sum? Note that the sum does **not** need to use each prime at least once.

    Answer: 13

    Solution: The primes available to us are 101, 103, 107, and 109.

    Since we need to sum up to an odd total, 2025, our sum must consist of an odd number of primes. We know we are going to use less than 21 numbers, since the minimum possible sum is too large: $2025<2121=21\cdot 101$. So, we check if we can use 19 or 17 numbers instead.

    Using 19 numbers is possible, since our maximum and minimum possible sums would be $19\cdot 109=2071>2025$ and $19\cdot 101=1919<2025$.

    Using 17 numbers is not, since our maximum sum would be $17\cdot 109=1853<2025$. So, we must have 19 numbers used.

    Since we know we will add up 19 numbers, we can simplify calculations by subtracting 101 from each number and $101\cdot 19=1919$ from the total sum. We need to sum up to $2025-1919=106$ using the numbers 0, 2, 6, and 8.

    Since we want to maximize the number of 109's (which are now 8's), we use the most possible 8's and then fill in the rest. This gives us $106=13(8)+1(2)+5(0)$. This means that the sum $13(109)+1(103)+5(101)=2025$, which uses $\boxed{13}$ 109's.

18. Let N_{16} be the answer to Problem 16 and N_{17} be the answer to Problem 17.

Given a right triangle with side lengths a, b, and c, we can transform it into a new right triangle using the following two-step transformation:

1. Remove one of the sides of the triangle.
2. Create a right triangle with the remaining two sides as the legs.

If we start with a right triangle with side lengths $1, 1, \sqrt{2}$, determine the minimum number of transformations needed to obtain a right triangle with legs of length $\sqrt[3]{N_{16}}$ and N_{17}.

Answer: 16

Solution: By squaring each side length, we can convert all the right triangles (e.g. $(1, 1, \sqrt{2}) \to (1, 1, 2)$) to become tuples of integers (X, Y, Z) where $X + Y = Z$. This avoids dealing with bothersome square roots.

The core of this solution is to <u>work backwards</u>.

In order to generate the tuple $(X, Y, X + Y)$, our previous tuple must have been from a smaller tuple (X, Y, n). But we know that n <u>must</u> be the difference between X and Y since it can't be $X + Y$. Thus, we can work backwards from $(N_{16}, N_{17}^2, N_{16} + N_{17}^2)$ until we get back to $(1, 1, 2)$, each time removing the largest number and introducing the difference of the smaller two.

With $N_{16} = 64$ and $N_{17} = 13$, our tuple is $(16, 169, 185)$. Now, work backwards to get the sequence of tuples $(16, 169, 185) \to (16, 153, 169) \to \cdots \to (16, 25, 41) \to (16, 9, 25) \to (7, 9, 16) \to (7, 2, 9) \to (5, 2, 7) \to (3, 2, 5) \to (1, 2, 3) \to (1, 1, 2)$ in $\boxed{16}$ steps.

19. Let N_{19} be the answer to this problem and N_{20} be the answer to Problem 20.

Square $ABCD$ has side length N_{20} and has points E and H inside such that $\overline{EH}$ is parallel to $\overline{AB}$. Point F is inside $ABCD$ such that the ray $\overrightarrow{EF}$ intersects $\overline{AB}$ at point F'. Point G is on $\overline{BC}$ such that $BG = \sqrt{N_{19}}$. Given that $\angle GFF' = 90°$ and $\angle FF'A = \angle FF'C = 45°$, determine the area of heptagon $AEFGCHD$.

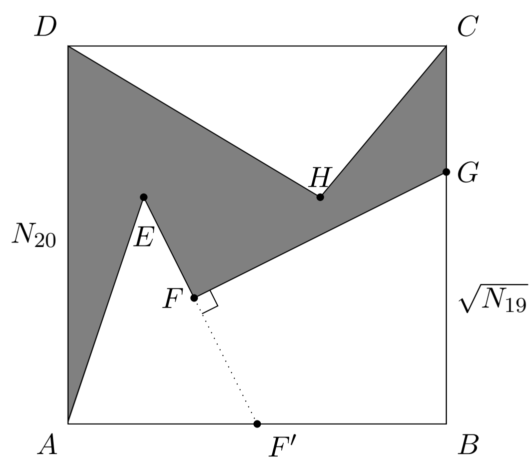


Diagram not drawn to scale.

Answer: $\mathbf{\frac{5}{8}}$

Solution: Looking at the angles $\angle FF'A = \angle FF'C = 45°$, it must be that $\angle AF'C = 90°$. However, the only spot on $\overline{AB}$ that forms an angle of $90°$ is when F' is the same point as B. Anywhere else would form an angle $\angle AF'C > 90°$.

So, our diagram is now:

[insert diagram]

Because E and H are the same distance from $\overline{AB}$, the combined area of triangles $\triangle CDH$ and $\triangle ABE$ is half of the area of square $ABCD$.

We also see that $\triangle BFG$ is a $45-45-90$ triangle, so $BF = FG = \frac{1}{\sqrt{2}}BG$ gives us the area $[BFG] = \frac{1}{2}(BF \cdot FG) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}BG \cdot \frac{1}{\sqrt{2}}BG\right) = \frac{1}{4}(BG^2) = \frac{1}{4}N_{19}$.

So, the area of heptagon $AEFGCHD$ is $[ABCD] - [CDH] - [ABE] - [BFG] = N_{20}^2 - \frac{1}{2}N_{20}^2 - \frac{1}{4}N_{19} = \frac{1}{2}N_{20}^2 - \frac{1}{4}N_{19}$.

Since this area is the answer to this problem,

$$N_{19} = \frac{1}{2}N_{20}^2 - \frac{1}{4}N_{19}$$

gives us

$$N_{19} = \frac{2}{5}N_{20}^2.$$

Plug in $N_{20} = \frac{5}{4}$ and we get $N_{19} = \boxed{\frac{5}{8}}$.

20. Let N_{19} be the answer to Problem 19 and N_{20} be the answer to this problem.

Two lines with equations $y = N_{20}x + a$ and $y = N_{19}x + b$ intersect at a point with x-coordinate equal to 1 in the coordinate plane. What is the value of N_{20} for which $b - a$ is maximized?

Answer: $\mathbf{\frac{5}{4}}$

Solution: $p(x) - q(x) = 0 \Rightarrow (N_{20} - N_{19}) \cdot x + (a - b) = 0$. When $x = 1$, we have $N_{20} - N_{19} = b - a$. Plugging in $N_{19} = \frac{2}{5}N_{20}^2$, we have:

$$N_{20} - \frac{2}{5}N_{20}^2 = b - a.$$

Since we are maximizing $b - a$, this is equivalent to maximize $N_{20} - \frac{2}{5}N_{20}^2$, which is a parabola that is concave down, using the vertex formula we can solve for $N_{20} = -\frac{1}{2 \cdot \left(-\frac{2}{5}\right)} = \boxed{\frac{5}{4}}$.