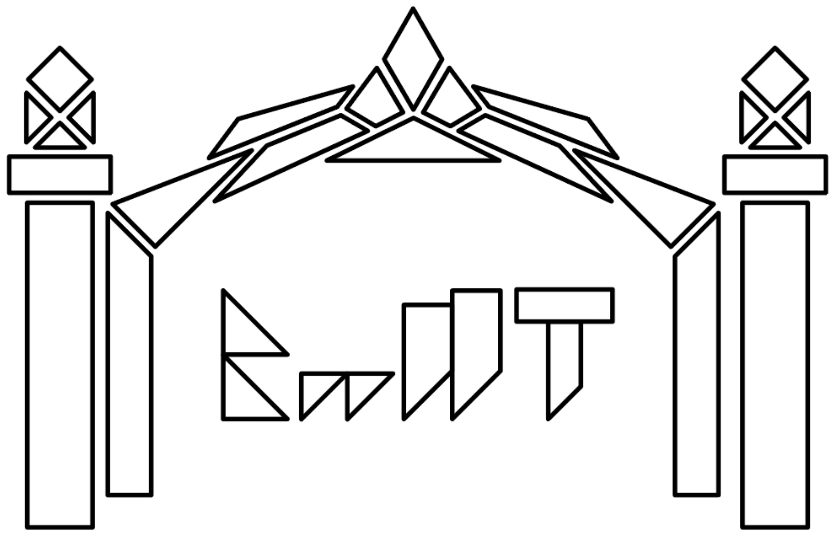
Berkeley mini Math Tournament 2024

Individual Round



April 14, 2024

Time limit: 60 minutes.

Instructions: This test contains 20 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators.

Answer format overview:

- Carry out any reasonable calculations. For instance, you should evaluate $\frac{1}{2} + \frac{1}{3}$, but you do not need to evaluate large powers such as 7^8.
- Write rational numbers in lowest terms. Decimals are also acceptable, provided they are exact. You may use constants such as π in your answers.
- Move all square factors outside radicals. For example, write $3\sqrt{7}$ instead of $\sqrt{63}$.
- Denominators do *not* need to be rationalized. Both $\frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{2}}$ are acceptable.
- Do not express an answer using a repeated sum or product.

1. Nikhil has 20 BMT shirts and 24 BmMT shirts. Wen takes half of his shirts, and then Arjun takes half of his remaining shirts. How many shirts does Nikhil have left?

2. What is the value of $2 + \frac{1}{2+\frac{1}{4}}$?

3. Rohit has a garden in the shape of a square with perimeter 2024. Youngmin has a lawn in the shape of a rectangle whose width is $\frac{1}{23}$ the width of Rohit's garden and whose length is $\frac{1}{22}$ the length of Rohit's garden. What is the area of Youngmin's lawn?

4. Anthony, Boris, and Carson are playing a game of frisbee! Anthony starts with the frisbee. On each turn, the player holding the frisbee throws it to someone else, after which the frisbee is either caught or dropped. If the frisbee is dropped, the nearest player picks it up. Throughout the game, Anthony throws the frisbee 8 times and catches it 6 times, Boris throws the frisbee 10 times and catches it 7 times, and Carson throws the frisbee 7 times and catches it 8 times. How many times is the frisbee dropped during the game?

5. Find the largest integer less than 2024 that leaves a remainder of 20 when divided by 24.

6. Find the sum of all prime numbers less than 50 that have only prime digits.

7. Let x and y be two numbers taken from the set $\{2^0, 2^1, 2^2, 2^3, \ldots, 2^{10}\}$ such that
$$xy - (x + y) = 104.$$
What is the value of $x + y$?

8. Thomas tosses a fair coin 3 times, and then Jefferson tosses the same coin 4 times. What is the probability that they flip exactly 3 heads in total?

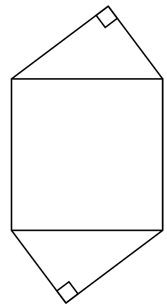
9. Let O_n be the sum of the first n positive odd integers, and let E_n be the sum of the first n positive even integers. For example, $E_1 = 2$ and $O_2 = 1 + 3 = 4$. What is the value of the product
$$\left(\frac{O_1}{E_1}\right)\left(\frac{O_2}{E_2}\right)\cdots\left(\frac{O_{2024}}{E_{2024}}\right)?$$

10. In triangle $\triangle ABC$, points D and E are chosen on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$. If $AE = BE$, $\angle BDE = 125°$, and $\angle ACB = 85°$, what is the value of $\angle BEC + \angle BAE$ in degrees?

11. Find the sum of all two-digit positive integers, x, such that x is a factor of 2024 and the units digit of x^{2024} is the same as the units digit of x.

12. What is the product of all real numbers, x, that satisfy
$$\frac{x^2 - 20x}{x^2 - 20x - 24} + \frac{x^2 - 20x + 24}{x^2 - 20x} = 2?$$

13. Consider a hexagon with consecutive side lengths of 8, 6, 10, 8, 6, and 10, formed by attaching two right triangles with legs of length 6 and 8 to the top and bottom of a square, as shown in the diagram below. Compute the largest distance between two points in this hexagon.



14. Kaity creates a list of all four-digit integers, $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$, with distinct digits such that $3000 < \underline{A}\,\underline{B}\,\underline{C}\,\underline{D} < 6000$, $\underline{A}\,\underline{B}\,\underline{C}$ is divisible by 3, $\underline{B}\,\underline{C}\,\underline{D}$ is divisible by 4, $\underline{D}\,\underline{C}\,\underline{B}$ is divisible by 5, and $\underline{C}\,\underline{B}\,\underline{A}$ is divisible by 6. Find the median of Kaity's list.

15. A *permutation* of a set of n integers is any arrangement of its elements in a specific order. For example, permutations of the list $(1, 2, 3)$ include $(1, 3, 2)$ and $(3, 2, 1)$. Let a permutation $(a_1, a_2, a_3, \dots, a_n)$ of the positive integers from 1 to n, inclusive, be a *katamari* if, for all $1 \leq i \leq n$, the inequality $a_i < 2 + a_1 + a_2 + a_3 + \cdots + a_{i-1}$ holds. For example, the permutation $(1, 2, 3, 4, 5, 7, 6)$ is a katamari but $(5, 4, 1, 2, 3)$ is not. Compute the number of permutations of $(1, 2, 3, \dots, 9)$ that are katamaris.

16. A positive integer is called *square-free* if it is not divisible by any perfect square greater than 1. Compute the sum of all positive integers, k, for which there exists a square-free positive integer, n, such that n^k has 64 divisors.

17. Let a sequence of integers, $a_1, a_2, \dots, a_n$, be *anti-consecutive* if $-2024 \leq a_1 \leq 2024$ and $a_i + a_{i+1} = i$ for all $1 \leq i \leq n - 1$. Over all anti-consecutive sequences of length 501, compute the sum of all possible values of a_{501} that are divisible by 3.

18. Let $\triangle ABC$ be an equilateral triangle with side length 1. Let O_a, O_b, and O_c be three congruent circles inside $\triangle ABC$ such that O_a is tangent to $\overline{CA}$ and $\overline{AB}$, O_b is tangent to $\overline{AB}$ and $\overline{BC}$, O_c is tangent to $\overline{BC}$ and $\overline{CA}$, and none of the circles intersect each other. Let O_m be a circle inside $\triangle ABC$ that is externally tangent to O_a, O_b, and O_c. What is the minimum possible sum of the areas of O_a and O_m?

19. Let S be the sum of the cubes of the divisors of 4200. Compute the last two digits of S.

20. Jonathan has 46 indistinguishable blue balls, 3 indistinguishable red balls, and a green ball in a bin. He continuously draws balls from the bin without replacement until he draws the green ball. For instance, Jonathan might draw a red ball, followed by two blue balls, another red ball, and then the green ball, completing the process. Compute the number of possible sequences of draws that are possible under these conditions.