1. There exist real numbers B, M, and T such that $B + M + T = 23$ and $B - M - T = 20$. Compute $M + T$.

   Answer: $\mathbf{\frac{3}{2}}$

   Solution: We can subtract the equations, which gives $2M + 2T = 3$, so $M + T = \boxed{\frac{3}{2}}$.

2. Kaity has a rectangular garden that measures 10 yards by 12 yards. Austin's triangular garden has side lengths 6 yards, 8 yards, and 10 yards. Compute the ratio of the area of Kaity's garden to the area of Austin's garden.

   Answer: 5

   Solution: Kaity's garden has area $10 \cdot 12 = 120$ square yards. Austin's garden is in the shape of a right triangle with legs 6 and 8 and hypotenuse 10, seeing as $6^2 + 8^2 = 10^2$. Thus, the area is simply $\frac{1}{2} \cdot 6 \cdot 8 = 24$ square yards. Finally, the ratio of the area of Kaity's garden to the area of Austin's garden is $\frac{120}{24} = \boxed{5}$.

3. Nikhil's mom and brother both have ages under 100 years that are perfect squares. His mom is 33 years older than his brother. Compute the sum of their ages.

   Answer: 65

   Solution 1: We can list all the perfect squares less than 100 as

   $$1, 4, 9, 16, 25, 36, 49, 64, 81.$$

   We find that the two numbers in this list that differ by 33 are 16 and 49. Thus, our answer is $49 + 16 = \boxed{65}$.

   Solution 2: We can also use difference of squares algebra. Let M^2 be his mom's age, and let B^2 be his brother's age, where $M, B > 0$. Then

   $$(M + B)(M - B) = M^2 - B^2 = 33.$$

   Thus, $M + B$ and $M - B$ are factors of 33. Because $M + B > M - B > 0$, we have two cases.

   - If $M + B = 33$ and $M - B = 1$, then solving gives $M = 17$, but then $M^2 > 100$.
   - If $M + B = 11$ and $M - B = 3$, then solving gives $M = 7$ and $B = 4$.

   The second case is the only possible case, so our answer is $M^2 + B^2 = 49 + 16 = \boxed{65}$.

4. Madison wants to arrange 3 identical blue books and 2 identical pink books on a shelf so that each book is next to at least one book of the **other** color. In how many ways can Madison arrange the books?

   Answer: 3

   Solution: One may try all of the possible arrangements, but it is faster to reduce the possibilities by applying the given restriction. Both the first and fifth books on the shelf are only adjacent to one book. So, the first and second books must be blue and pink in some order, and the fourth and fifth books must be blue and pink in some order. But the only book remaining is blue, so the third book is blue, which implies that the second or fourth book must be pink. Thus, the valid arrangements are

   $$BPBBP, \qquad PBBPB, \qquad \text{and} \qquad BPBPB,$$

   for a total of $\boxed{3}$ arrangements.

5. Two friends, Anna and Bruno, are biking together at the **same** initial speed from school to the mall, which is 6 miles away. Suddenly, 1 mile in, Anna realizes that she forgot her calculator at school. If she bikes 4 miles per hour faster than her initial speed, she could head back to school and still reach the mall at the same time as Bruno, assuming Bruno continues biking towards the mall at their initial speed. In miles per hour, what is Anna and Bruno's **initial** speed, before Anna has changed her speed? (Assume that the rate at which Anna and Bruno bike is constant.)

   Answer: 10

   Solution: In general, we know $d = rt$, where d is a distance traveled, r is the rate (or speed) traveled, and t is the time traveled. Thus, we can represent our situation as a simple algebraic equation in terms of their intial speed x. We are using the fact that $t = \frac{d}{r}$ and equating their times: once they are 1 mile in, Anna must travel $6+1$ miles at a rate of $x+4$, and Bruno must travel $6-1$ miles at a rate of x. Thus,

$$\begin{aligned} \frac{6+1}{x+4} &= \frac{6-1}{x} \\ \frac{7}{x+4} &= \frac{5}{x} \\ 5(x+4) &= 7x \\ 5x+20 &= 7x \\ 2x &= 20 \\ x &= 10. \end{aligned}$$

   This means their initial shared speed is $\boxed{10}$ miles per hour.

6. Let a number be "almost-perfect" if the sum of its digits is 28. Compute the sum of the third smallest and third largest almost-perfect 4-digit positive integers.

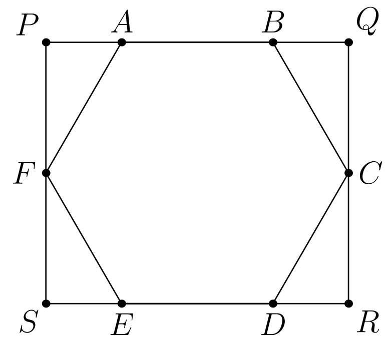
   Answer: 12962

   Solution: We have two computations.

   - The smallest possible almost-perfect 4-digit integer is 1999. To get the second smallest, we must change the 1 to a 2, and we can change the second 9 to an 8, which gives 2899. We can simply rearrange the 899 at the end to see that the third smallest is 2989.
   - Similarly, the third largest can be obtained by starting with the largest, which is 9991. Arguing as before, the second largest is 9982, and the third largest is therefore 9973.

   Thus, the answer is $2989 + 9973 = \boxed{12962}$.

7. Regular hexagon $ABCDEF$ is contained in rectangle $PQRS$ such that line $\overline{AB}$ lies on line $\overline{PQ}$, point C lies on line $\overline{QR}$, line $\overline{DE}$ lies on line $\overline{RS}$, and point F lies on line $\overline{SP}$. Given that $PQ = 4$, compute the perimeter of $AQCDSF$.



Answer: $10 + 2\sqrt{3}$

Solution: To begin, we compute the side length s of $ABCDEF$. Because triangle $\triangle BCQ$ is a 30-60-90 triangle, $BQ = \frac{1}{2}BC = \frac{s}{2}$, and $CQ = \frac{\sqrt{3}}{2}BC = \frac{s\sqrt{3}}{2}$. Also, by symmetry, $PA = \frac{s}{2}$, so

$$4 = PQ = PA + AB + BQ = 2s$$

implies $s = 2$. Finally, using symmetry, the perimeter of $AQCDSF$ is

$$2(AB + BQ + QC + CD) = 2\left(\frac{3s}{2} + \frac{s\sqrt{3}}{2} + s\right) = s(5 + \sqrt{3}) = \boxed{10 + 2\sqrt{3}}.$$

8. Compute the number of ordered pairs (m, n), where m and n are relatively prime positive integers and $mn = 2520$. (Note that positive integers x and y are relatively prime if they share no common divisors other than 1. For example, this means that 1 is relatively prime to every positive integer.)

   Answer: 16

   Solution: First, we prime factorize $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. For each of these four primes, exactly one of m and n can be divisible by the prime. Moreover, if m is divisible by prime p, then because n can have no factors of p, we see m must have the same number of p factors as 2520 does. Thus, we are computing the number of ways to assign each of 2^3, 3^2, 5, and 7 to one of m or n, so the answer is $2^4 = \boxed{16}$.

9. A geometric sequence with more than two terms has first term x, last term 2023, and common ratio y, where x and y are both positive integers greater than 1. An arithmetic sequence with a finite number of terms has first term x and common difference y. Also, of all arithmetic sequences with first term x, common difference y, and no terms exceeding 2023, this sequence is the longest. What is the last term of the arithmetic sequence?

   Answer: 2013

   Solution: Let the geometric sequence have length $n \geq 3$. Then the last term of the geometric sequence satisfies $xy^{n-1} = 2023$. In particular, y^2 divides $2023 = 7 \cdot 17^2$, so $y = 17$. Because x is an integer, we must have $x = 7$ and $n = 3$.

   Now, let the arithmetic sequence have length k. The kth term of the arithmetic sequence is $7 + 17(k-1) \leq 2023$, so

$$k \leq \left\lfloor \frac{2023 - 7}{17} + 1 \right\rfloor = 119.$$

   The last term in the arithmetic sequence is thus $7 + 17 \cdot (119 - 1) = \boxed{2013}$.

10. Andrew is playing a game where he must choose three slips, uniformly at random and without replacement, from a jar that has nine slips labeled 1 through 9. He wins if the sum of the three chosen numbers is divisible by 3 and one of the numbers is 1. What is the probability Andrew wins?

    Answer: $\frac{5}{42}$

    Solution: Because he is choosing without replacement, we know the digits must be distinct. There are $\binom{9}{3} = 84$ ways to choose 3 distinct integers. We are told that one of the slips must be a 1, and the sum of the slips must be divisible by 3. They need to be distinct, so the minimum sum is $1 + 2 + 3 = 6$. Analogously, the maximum sum is $1 + 8 + 9 = 18$. We now do casework.

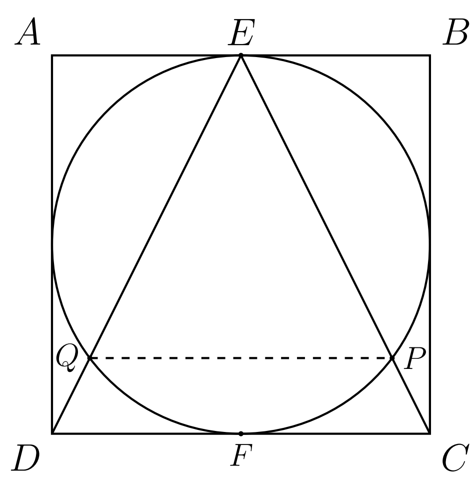
    - To achieve a sum of 6, we need the chosen slips to be $\{1, 2, 3\}$.
    - To achieve a sum of 9, we can have either $\{1, 2, 6\}$ or $\{1, 3, 5\}$.
    - To achieve a sum of 12, we can have $\{1, 2, 9\}$, $\{1, 3, 8\}$, $\{1, 4, 7\}$, or $\{1, 5, 6\}$.

    To continue, note that there is a symmetry: a choice of slips $\{1, a, c\}$ with sum $1+a+c$ yields a choice of slips $\{1, 11-a, 11-c\}$ with sum $24-(1+a+c)$. Thus, the number of ways to achieve a sum of 15 is the same as the number of ways for 9, and the number of ways to achieve a sum of 18 is the same as the number of ways for 6. Totaling, there are $1+2+4+2+1 = 10$ possibilities. Now, we do not care about order the slips are drawn because the numbers are distinct, so our answer is $\frac{10}{84} = \boxed{\frac{5}{42}}$.

11. Circle O is inscribed in square $ABCD$. Let E be the point where O meets line segment $\overline{AB}$. Line segments $\overline{EC}$ and $\overline{ED}$ intersect O at points P and Q, respectively. Compute the ratio of the area of triangle $\triangle EPQ$ to the area of triangle $\triangle ECD$.

    Answer: $\frac{16}{25}$

    Solution: Here is a diagram.



    Notice that if we are able to find the ratio between EP and EC, we can find the ratio of the triangles' areas by squaring. Suppose, for convenience, that the side length of $ABCD$ is 2. Applying the Pythagorean theorem to triangle $\triangle EBC$ yields that $EC = \sqrt{1^2 + 2^2} = \sqrt{5}$.

As in the diagram, denote the intersection of O with line segment $\overline{CD}$ as F. Constructing line segment $\overline{PF}$ yields a set of similar triangles, most usefully that $\triangle CPF \sim \triangle EBC$. Applying similarity, we know that

$$\frac{CP}{CF} = \frac{EB}{EC},$$

so $CP = CF \cdot \frac{EB}{EC} = 1 \cdot \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$.

Now, the length of side $\overline{EP}$ of triangle $\triangle EPQ$ is therefore equal to $\sqrt{5} - \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{4}{\sqrt{5}}$. Thus, the ratio of the area of triangle $\triangle EPQ$ to the area of triangle $\triangle ECD$ is

$$\left(\frac{EP}{EC}\right)^2 = \left(\frac{4/\sqrt{5}}{\sqrt{5}}\right)^2 = \boxed{\frac{16}{25}}.$$

12. Define a recursive sequence by $a_1 = \frac{1}{2}$ and $a_2 = 1$, and

$$a_n = \frac{1 + a_{n-1}}{a_{n-2}}$$

for $n \geq 3$. The product

$$a_1 a_2 a_3 \ldots a_{2023}$$

can be expressed in the form $a^b \cdot c^d \cdot e^f$, where a, b, c, d, e, and f are positive (not necessarily distinct) integers, and a, c, and e are prime. Compute $a + b + c + d + e + f$.

Answer: 819

Solution: A key observation is that the recursive sequence of this form cycles every five terms in general regardless of the starting values. We see this by letting $a_1 = x$ and $a_2 = y$. Then the sequence becomes the following:

n	1	2	3	4	5	6	7	$\cdots$
a_n	x	y	$\frac{1+y}{x}$	$\frac{x+y+1}{xy}$	$\frac{x+1}{y}$	x	y	$\cdots$

Using this, we can compute that our first 5 terms are $\frac{1}{2}, 1, 4, 5, \frac{3}{2}$, whose product is 15. Since this sequence cycles every five terms, we need to find the quotient and remainder of 2023 divided by 5, which are 404 and 3 respectively. Hence,

$$a_1 a_2 a_3 \cdots a_{2023} = (a_1 a_2 a_3 a_4 a_5)^{404} \cdot a_1 a_2 a_3 = 15^{404} \cdot 2 = 2^1 \cdot 3^{404} \cdot 5^{404}.$$

Thus, our desired sum is $2 + 1 + 3 + 404 + 5 + 404 = \boxed{819}$.

13. An increasing sequence of 3-digit positive integers satisfies the following properties:

- Each number is a multiple of 2, 3, or 5.
- Adjacent numbers differ by only one digit and are relatively prime. (Note that positive integers x and y are relatively prime if they share no common divisors other than 1.)

What is the maximum possible length of the sequence?

Answer: 7

Solution: Our main claim is that only the units digit may change in the sequence. Each number in the sequence is either a multiple of 3, or has a units digit of 0, 2, 4, 5, 6, or 8. We therefore have the following two cases:

- If a term in the sequence is divisible by 2 or by 5, then the next term cannot be divisible by 2 or by 5 (respectively), so the units digit must change and will thus be the only digit that changes.
- Otherwise, a term in the sequence is divisible by 3 but not by 2 nor by 5, so we see that the next term must be divisible by 2 or by 5, which again requires adjusting the units digit.

Now, because only the units digit may change, we know that the multiples of 3 may end in 0, 3, 6, 9; or 1, 4, 7; or 2, 5, 8. We now do casework.

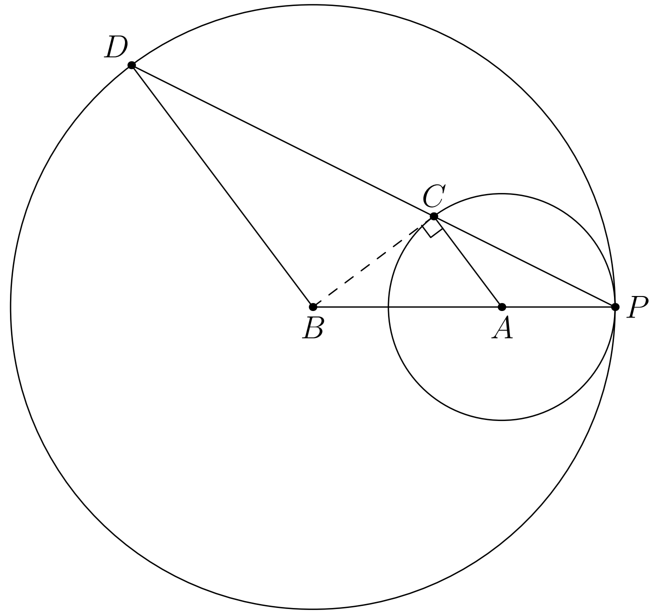
- Choosing the sequence 0, 3, 6, 9, we would like to add multiples of 2 and 5. The digits we have to work with are $0, 2, 3, 4, 5, 6, 8, 9$, but we cannot have both 0 and 2 nor both 6 and 8, so our sequence has a length of at most six.
- Choosing the sequence 1, 4, 7, we again would like to add multiples of 2 and 5. The digits we have to work with are $0, 1, 2, 4, 5, 6, 7, 8$, but we cannot have both 2 and 4, so our sequence has a length of at most seven.
- Choosing the sequence 2, 5, 8, we again add multiples of 2 and 5. The digits we have to work with are $0, 2, 4, 5, 6, 8$, so our sequence has a length of at most six.

Thus, we see that the longest possible length is $\boxed{7}$, in the second case. One example of such a maximal sequence of 3-digit numbers satisfying the property is $200, 201, 202, 205, 206, 207, 208$.

14. Circles O_A and O_B with centers A and B, respectively, have radii 3 and 8, respectively, and are internally tangent to each other at point P. Point C is on circle O_A such that line $\overline{BC}$ is tangent to circle O_A. Extend line $\overline{PC}$ to intersect circle O_B at point $D \neq P$. Compute CD.

Answer: $\mathbf{4\sqrt{5}}$

Solution: Note that triangles $\triangle PAC$ and $\triangle PBD$ are isosceles triangles, since $PA = AC$ and $PB = BD$ (they're radii of their respective circles).



Because $\angle ACP = \angle APC = \angle BPD = \angle BDP$, we see $\overline{AC} \parallel \overline{BD}$. Because line $\overline{BC}$ is a tangent to O_A, we see $\overline{BC} \perp \overline{AC}$, so $\overline{BD} \perp \overline{BC}$. Now all that remains is to compute the necessary lengths.

We have $AB = BP - AP = 8 - 3 = 5$, so $BC = \sqrt{5^2 - 3^2} = 4$. Therefore, $CD = \sqrt{4^2 + 8^2} = \boxed{4\sqrt{5}}$.

15. Compute the product of all real solutions x to the equation $x^2 + 20x - 23 = 2\sqrt{x^2 + 20x + 1}$.

Answer: −35

Solution: Set $y = x^2 + 20x - 23$, so we're solving

$$y = 2\sqrt{y + 24}.$$

Squaring gives $y^2 = 4(y + 24)$, which rearranges into

$$0 = y^2 - 4y - 96 = (y - 12)(y + 8).$$

We can discard the solution $y = -8$ because $2\sqrt{y + 24}$ must be non-negative. Thus, we substitute $y = 12$ so that we are solving

$$x^2 + 20x - 23 = 12.$$

Equivalently, we want $x^2 + 20x - 35 = 0$, which produces the same solutions for x as our original equation. The discriminant of this quadratic is $20^2 - 4(1)(-35) > 0$, so the roots of this equation are distinct real numbers. By Vieta's formulae, the product of the roots of this quadratic is $\boxed{-35}$.

16. Compute the number of divisors of 729, 000, 000 that are perfect powers. (A perfect power is an integer that can be written in the form a^b, where a and b are positive integers and $b > 1$.)

Answer: 90

Solution: We have $729,000,000 = 2^6 \cdot 3^6 \cdot 5^6$, so every divisor d is of the form $2^a \cdot 3^b \cdot 5^c$ for integers $0 \leq a, b, c \leq 6$. Thus, if $d = x^y$ is a perfect power with $y > 1$, we see that $2 \leq y \leq 6$, so we have the following cases for y:

- If d is a perfect square, then $a, b, c \in \{0, 2, 4, 6\}$. There are $4^3 = 64$ possibilities for a, b, c.
- If d is a perfect cube, then $a, b, c \in \{0, 3, 6\}$. There are $3^3 = 27$ possibilities for a, b, c.
- If d is a perfect fifth power, then $a, b, c \in \{0, 5\}$. There are $2^3 = 8$ possibilities for a, b, c.

While we have a total count of $64 + 27 + 8 = 99$, there is some overcounting. Namely, we've doubled counted sixth powers as both perfect squares and perfect cubes. If d is a sixth power, then $a, b, c \in \{0, 6\}$. There are $2^3 = 8$ possibilities for a, b, c, so our count is adjusted to $99 - 8 = 91$. Finally, 1 was counted three times, once as a perfect square, once as a perfect cube, and once as a perfect fifth power, and was subtracted once as a perfect sixth power, so we subtract 1 another time from our count, yielding a total of $91 - 1 = \boxed{90}$.

17. The arithmetic mean of two positive integers x and y, each less than 100, is 4 more than their geometric mean. Given $x > y$, compute the sum of all possible values for $x + y$. (Note that the geometric mean of x and y is defined to be $\sqrt{xy}$.)

Answer: 380

Solution: We have $\frac{x+y}{2} = \sqrt{xy} + 4$. We can rewrite this as $x + y = 2\sqrt{xy} + 8$. Let $a = \sqrt{x}$ and $b = \sqrt{y}$. Then $a^2 + b^2 = 2ab + 8$, which rearranges to $(a - b)^2 = 8$. This implies

$$a - b = \sqrt{8} = 2\sqrt{2}.$$

We see $a - b = -2\sqrt{2}$ is not possible since

$$x > y \implies \sqrt{a} > \sqrt{b} \implies a > b \implies a - b > 0.$$

We want to show that $b = k\sqrt{2}$ for some integer k. Indeed, we write

$$x = a^2 = \left(b + 2\sqrt{2}\right)^2 = b^2 + 8 + 4b\sqrt{2}.$$

Rearranging, we see $4b\sqrt{2}$ must be equal to some integer c, so $32b^2 = c^2$. Thus, c^2 is divisible by 8, which means we can write $c = 8k$, yielding $b^2 = 2k^2$. Thus, b is of the form $b = \pm k\sqrt{2}$ as desired.

In total, we claim our possible solution pairs for (a, b) are

$$\left(3\sqrt{2}, \sqrt{2}\right), \quad \left(4\sqrt{2}, 2\sqrt{2}\right), \quad \left(5\sqrt{2}, 3\sqrt{2}\right), \quad \left(6\sqrt{2}, 4\sqrt{2}\right), \quad \text{and} \quad \left(7\sqrt{2}, 5\sqrt{2}\right).$$

Notably, we must have $b > 0$ because $y = b^2$ is a positive integer. Further, because $(8\sqrt{2})^2 = 128 > 100$, we can stop after this fifth solution pair. Our possibilites for the two integers (x, y) are therefore the pairs

$$(18, 2), \quad (32, 8), \quad (50, 18), \quad (72, 32), \quad \text{and} \quad (98, 50).$$

The sum of all the possible values for $x + y$ here is $20 + 40 + 68 + 104 + 148 = \boxed{380}$.

18. Ankit and Richard are playing a game. Ankit repeatedly writes the digits 2, 0, 2, 3, in that order, from left to right on a board until Richard tells him to stop. Richard wins if the resulting number, interpreted as a base-10 integer, is divisible by as many positive integers less than or equal to 12 as possible. For example, if Richard stops Ankit after 7 digits have been written, the number would be 2023202, which is divisible by 1 and 2. Richard wants to win the game as early as possible. Assuming Ankit must write at least one digit, after how many digits should Richard stop Ankit?

Answer: 258

Solution: Notice that 2023 is divisible by 7, and no other prefix of 2023 is divisible by 7. Hence, for the resulting number to be divisible by 7, we need to stop at the end of a 2023. However, doing so would avoid divisibility by numbers such as 2, 4, 5, etc., which we would achieve if we stopped at 20 instead. In fact, ending with 320 automatically gives us divisibility by $1, 2, 4, 5, 8, 10$; conversely, to be divisible by 8 and 10, we must end with 320. Next up, we are looking to also achieve divisibility by 9 and 11, which automatically gives us divisibility by $3, 6, 12$ if we end with 320. In total, we will be divisible by all positive integers less than or equal to 12 except 7, which is clearly optimal, as we have shown.

Suppose Ankit writes n instances of 2023 before writing the last 20. For the number to be divisible by 9, Richard must ensure that the sum of digits, $7n + 2$, is divisible by 9, which is equivalent to $7n - 7$ and thus $n - 1$ being divisible by 9. Similarly, for the number to be divisible by 11, Richard must ensure that the alternating sum of digits, $n + 2$, is divisible by 11. The smallest positive integer n such that

$$\begin{cases} n - 1 \equiv 0 \pmod{9}, \\ n + 2 \equiv 0 \pmod{11}, \end{cases}$$

is $n = 64$, which gives us an answer of $64 \cdot 4 + 2 = \boxed{258}$.

19. Eight chairs are set around a circular table. Among these chairs, two are red, two are blue, two are green, and two are yellow. Chairs that are the same color are identical. If rotations and reflections of arrangements of chairs are considered distinct, how many arrangements of chairs satisfy the property that each pair of adjacent chairs are different colors?

    Answer: 744

    Solution: We conduct a count by the Principle of Inclusion-Exclusion.

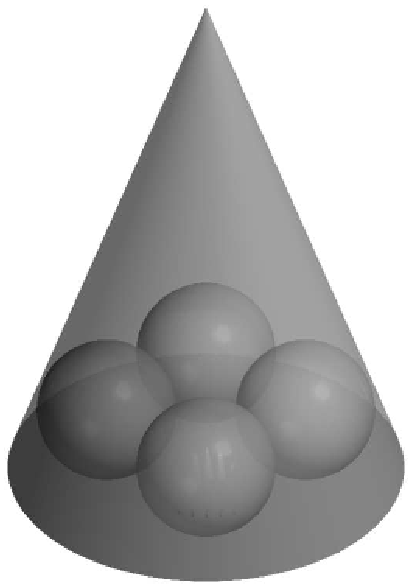
    - Ignoring the different-color adjacency condition, there are $\frac{8!}{2!2!2!2!} = 2520$ arrangements of chairs.
    - We now subtract off each of the arrangements that have a specific chair pair adjacent for each of the 4 colors: we "glue" the pair of chairs together and arrange them. For each of the 4 colors, there are 8 ways to place the glued pair of chairs, and $\frac{6!}{2!2!2!} = 90$ ways to place the other chairs in the remaining spaces, for a total of $4 \cdot 8 \cdot 90 = 2880$ cases to subtract.
    - We now add back each of the arrangements which have 2 specific chair pairs adjacent, for each combination of 2 colors. Again, we "glue" each pair of chairs together. Then for each of the $\binom{4}{2} = 6$ pairs of colors, there are $8 \cdot 5 = 40$ ways to place the two pairs of glued chairs, and $\frac{4!}{2!2!} = 6$ ways to place the other chairs, for a total of $6 \cdot 40 \cdot 6 = 1440$ cases to add back.
    - We then subtract off each of the arrangements that have 3 specific chair pairs adjacent, for each combination of 3 colors. Again, we "glue" each pair of chairs together. There are $\binom{4}{3} = 4$ combinations of 3 colors whose chairs will be glued together. This time, the easier approach is to place the pair of unglued chairs first, which may be done in $\frac{8 \cdot 4}{2} = 16$ ways because we need to ensure that the spaces created between them have an even number of spots. After that, there are $3! = 6$ ways to place the pairs of glued chairs in the remaining spots, for a total of $4 \cdot 16 \cdot 6 = 384$ ways to subtract off.
    - Finally, we add back the case where we have each pair of chairs adjacent. There are 8 ways to place the pair of red chairs, and $3! = 6$ ways to place the other pairs, so there are $8 \cdot 6 = 48$ ways that we need to add back.

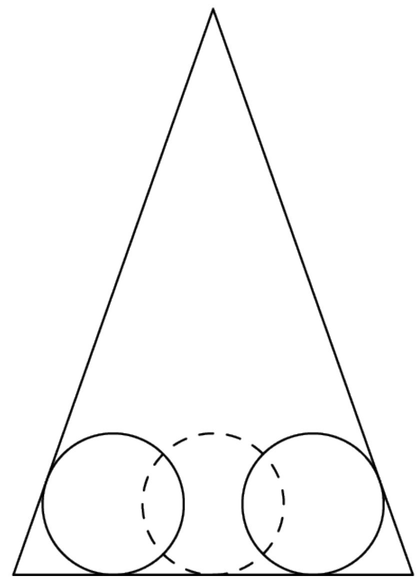
    Our final count is $2520 - 2880 + 1440 - 384 + 48 = \boxed{744}$.

20. Four congruent spheres are placed inside a right-circular cone such that they are all tangent to the base and the lateral face of the cone, and each sphere is tangent to exactly two other spheres. If the radius of the cone is 1 and the height of the cone is $2\sqrt{2}$, what is the radius of one of the spheres?

    Answer: $\mathbf{\frac{\sqrt{2}}{4}}$

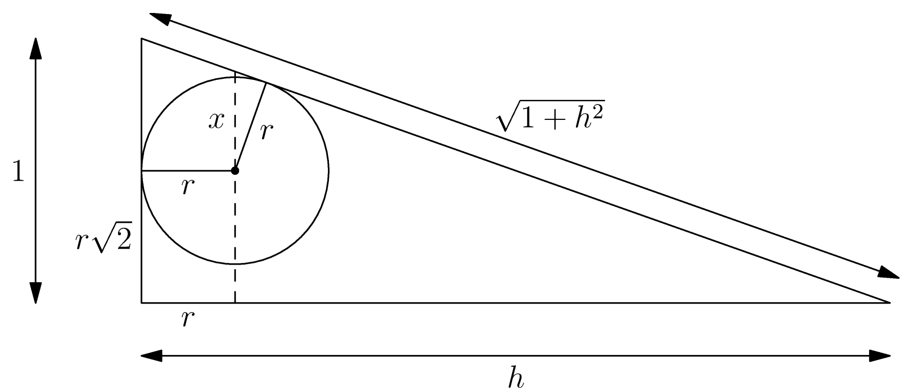
    Solution: Consider the following diagrams.





Take the cross section that passes through the center of two spheres not tangent to each other and perpendicular to the base of the cone, as shown in the diagram above.

Now, we rotate, halve, and label the triangle into the following diagram:



Explicitly, r is the radius of a sphere, $h = 2\sqrt{2}$ is the height of the cone, and x is the length of the dashed segment from the lateral face to the center of the sphere. Note that, since the centers of the spheres form a square of side length $2r$, the distance from the center of the sphere to the altitude of the cone is $r\sqrt{2}$. Further, let ℓ be the length of the entire dashed line segment. On one hand, by similar triangles, note that $\frac{x}{r} = \frac{\sqrt{1+h^2}}{h}$, so the length of the dashed line segment is

$$\ell = x + r\sqrt{2} = r \cdot \frac{\sqrt{1+h^2}}{h} + r\sqrt{2}.$$

On the other hand, by similar triangles again, we see $\frac{\ell}{1} = \frac{h-r}{h}$, so we compute

$$\frac{h-r}{h} = \ell = r \cdot \frac{\sqrt{1+h^2}}{h} + r\sqrt{2}.$$

Clearing denominators and rearranging, we find

$$h = r\left(1 + \sqrt{1+h^2} + h\sqrt{2}\right),$$

so

$$r = \frac{h}{1 + \sqrt{1+h^2} + h\sqrt{2}}.$$

Plugging in $h = 2\sqrt{2}$, we get $r = \frac{2\sqrt{2}}{8} = \boxed{\frac{\sqrt{2}}{4}}$.