1. What is the area of a triangle with side lengths 6, 8, and 10?

   Answer: 24

   Solution: Note that this triangle is right since $6^2 + 8^2 = 10^2$. Thus, the area is $\frac{6 \cdot 8}{2} = \boxed{24}$.

2. Let $f(n) = \sqrt{n}$. If $f(f(f(n))) = 2$, compute n.

   Answer: 256

   Solution: Square rooting and squaring are inverses. Thus, squaring both sides of the given equation three times to undo all of the fs, we have $n = ((2^2)^2)^2 = 2^8 = \boxed{256}$.

3. Anton is buying AguaFina water bottles. Each bottle costs 14 dollars, and Anton buys at least one water bottle. The number of dollars that Anton spends on AguaFina water bottles is a multiple of 10. What is the least number of water bottles he can buy?

   Answer: 5

   Solution: The number of dollars that Anton spends must be a multiple of the cost of a bottle, so the number of dollars spent is a multiple of 10 and 14. The least common multiple of 10 and 14 is $2 \cdot 5 \cdot 7 = 70$, so the least number of bottles that Anton can buy is $\frac{70}{14} = \boxed{5}$.

4. Alex flips 3 fair coins in a row. The probability that the first and last flips are the same can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

   Answer: 3

   Solution: No matter what the first flip is, the chance that the last flip is the same side is $\frac{1}{2}$, since the flipping of fair coins is independent. Thus, the probability is $\frac{1}{2}$, and our answer is $\boxed{3}$.

5. How many prime numbers p satisfy the property that $p^2 - 1$ is *not* a multiple of 6?

   Answer: 2

   Solution 1: We can factor $p^2 - 1$ in the form $(p - 1)(p + 1)$. Either 2 is not a factor of this product, or 3 is not a factor. If 2 is not a factor of this product, then $p - 1$ and $p + 1$ must both be odd, so p is even. The only even prime is $p = 2$.

   If 3 is not a factor of this product, then p cannot leave a remainder of 1 or 2 when divided by 3, otherwise the product $(p - 1)(p + 1)$ would have a factor of 3. Thus, p must be divisible by 3. The only such prime is $p = 3$.

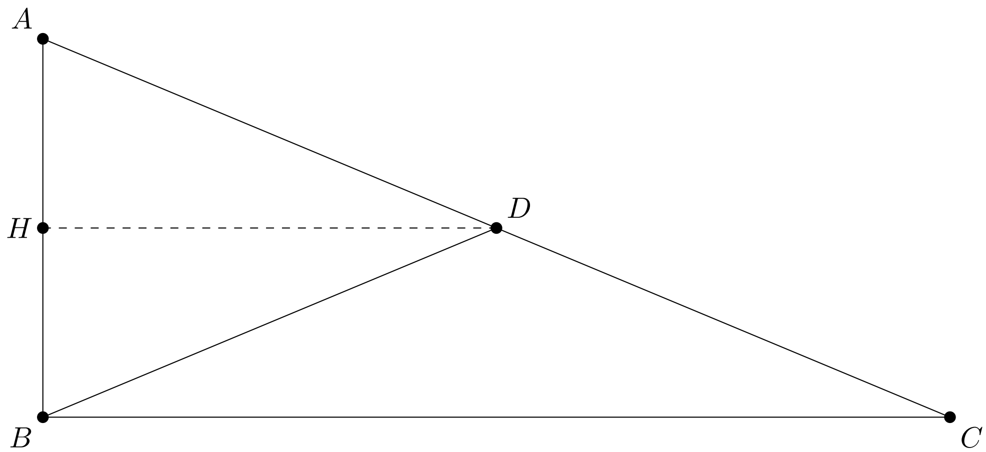
   Thus, there are a total of $\boxed{2}$ primes such that $p^2 - 1$ is not divisible by 6.

   Solution 2: Note that if $p > 3$, p must be odd and not divisible by 3, so p has a remainder of 1 or 5 when divided by 6, and so p^2 has a remainder of 1 when divided by 6. Thus, $p^2 - 1$ is a multiple of 6 when $p > 3$. This leaves $p = 2$ and $p = 3$, both of which satisfy the property that $p^2 - 1$ is not a multiple of 6, for a total of $\boxed{2}$ primes.

6. In right triangle $\triangle ABC$ with $AB = 5$, $BC = 12$, and $CA = 13$, point D lies on $\overline{CA}$ such that $AD = BD$. The length of CD can then be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

   Answer: 15

   Solution: Let the altitude of $\triangle ADB$ from D intersect $\overline{AB}$ at H. $\triangle ADB$ is isosceles since $AD = DB$, so H is the midpoint of $\overline{AB}$. Combined with the fact that $\overline{DH} \parallel \overline{BC}$ because they are both perpendicular to $\overline{AB}$, we see that D is the midpoint of $\overline{AC}$. Thus, $CD = \frac{13}{2}$, and our desired answer is $\boxed{15}$.



7. Vivienne is deciding on what courses to take for Spring 2021, and she must choose from four math courses, three computer science courses, and five English courses. Vivienne decides that she will take one English course and two additional courses that are either computer science or math, and the order of the courses does not matter. How many choices does Vivienne have?

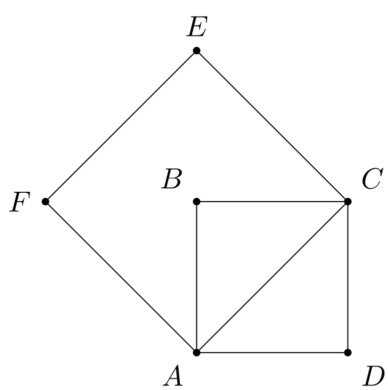
   Answer: 105

   Solution: There are $\binom{4+3}{2} = 21$ choices for the math/computer science courses and $\binom{5}{1} = 5$ choices for the English course, so there are $21 \cdot 5 = \boxed{105}$ total choices.

8. Square $ABCD$ has side length 2. Square $ACEF$ is drawn such that B lies inside square $ACEF$. Compute the area of pentagon $AFECD$.

   Answer: 10

   Solution:



   By symmetry, B is the center of the square, so the areas of triangles $\triangle ABC$, $\triangle CBE$, $\triangle EBF$, and $\triangle FBA$ are all the same. Moreover, the area of $\triangle ACD$ is the same as that of $\triangle ABC$. Thus, the area of $AFECD$ is five times the area of $\triangle ABC$, or $5 \cdot \frac{1}{2} \cdot 2 \cdot 2 = \boxed{10}$.

9. At the Boba Math Tournament, the Blackberry Milk Team has answered 4 out of the first 10 questions on the Boba Round correctly. If they answer all p remaining questions correctly, they will have answered exactly $\frac{9p}{5}\%$ of the questions correctly in total. How many questions are on the Boba Round?

Answer: 60

Solution: We can set up the equation $\frac{4+p}{10+p} = \frac{9p}{500}$. Solving this equation gives $p = 50$, and thus there are $\boxed{60}$ problems on the Boba Round in total.

10. The sum of two positive integers is 2021 less than their product. If one of them is a perfect square, compute the sum of the two numbers.

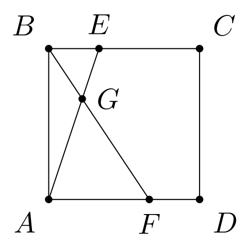
Answer: 679

Solution: Let the two numbers be m^2 and n, where m and n are positive integers. Then $m^2n - m^2 - n = 2021$, which gives us $(m^2 - 1)(n - 1) = 2022$. Since $2022 = 2 \times 3 \times 337$, the only possible ordered pair (m, n) that satisfies this equation is $(2, 675)$, so $m^2 + n = 2^2 + 675 = \boxed{679}$.

11. Points E and F lie on edges $\overline{BC}$ and $\overline{DA}$ of unit square $ABCD$, respectively, such that $BE = \frac{1}{3}$ and $DF = \frac{1}{3}$. Line segments $\overline{AE}$ and $\overline{BF}$ intersect at point G. The area of triangle EFG can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: 10

Solution:



Observe that triangles $\triangle AFG$ and $\triangle BEG$ are similar. Since $BE = \frac{1}{2}AF$, $FG = 2GB$, so the height of triangle $\triangle ABG$ is $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$, making its area $\frac{1}{9}$. We obtain the area of $\triangle EFG$ by subtracting the sum of areas of $\triangle ABF$ and $\triangle ABE$ from the area of trapezoid $ABEF$, then adding the area of $\triangle ABG$. Thus, the area of $\triangle EFG$ is $(\frac{1}{2}(\frac{1}{3} + \frac{2}{3})) - \frac{1}{3} - \frac{1}{6} + \frac{1}{9} = \frac{1}{9}$, and that our desired answer is $\boxed{10}$.

12. Compute the number of positive integers $n \leq 2020$ for which n^{k+1} is a factor of $(1+2+3+\cdots+n)^k$ for some positive integer k.

Answer: 1

Solution: Observe that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$, so n^{k+1} must be a factor of $\frac{n^k(n+1)^k}{2^k}$. Thus, we require that n be a factor of $\frac{(n+1)^k}{2^k}$, which implies that $\frac{(n+1)^k}{2^k \cdot n}$ is an integer. If n was even, the numerator would not be a multiple of 2, so, n must be odd. However, $(n+1)^k$ is not a multiple of n for any n other than 1, since $\gcd(n, n+1) = 1$. Then there is only $\boxed{1}$ solution: $n = 1$.

13. How many permutations of 123456 are divisible by their last digit? For instance, 123456 is divisible by 6, but 561234 is not divisible by 4.

Answer: 648

Solution: For all $d \neq 4$, any permutation ending in d is divisible by d. In particular, $d = 1$ is immediate, $d = 2$ works because any number ending in 2 is even, $d = 5$ works because any number ending in 5 is a multiple of 5, and $d = 3$ and $d = 6$ result from any permutation of the number being a multiple of 3 (as the digits sum to 21, a multiple of 3), and for $d = 6$, also because 6 is even. Once the last digit is chosen for each of these five cases, there are 5! to arrange the other digits, for a total of $5 \cdot 5!$ permutations. However, for $d = 4$, the last two digits must be a multiple of 4, which only happens when the second-to-last digit is 2 or 6. This gives an additional $2 \cdot 4! = 48$ permutations, for a total of $5 \cdot 5! + 48 = \boxed{648}$ permutations.

14. Compute the sum of all possible integer values for n such that $n^2 - 2n - 120$ is a positive prime number.

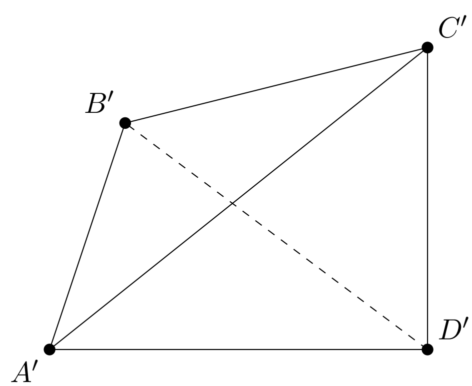
    Answer: 2

    Solution: Since we can factor $n^2 - 2n - 120 = (n - 12)(n + 10)$, for this to be prime, one of the factors must be ± 1. This gives that $n - 12 = \pm 1$ or $n + 10 = \pm 1$, which gives the possible values of n as $-11, -9, 11$, or 13. Plugging these values in for n gives $23, -21, -21$, and 23, respectively, so the only values of n that return a positive prime number are $n = -11$ and $n = 13$. Thus, the sum of all possible values is $-11 + 13 = \boxed{2}$.

15. Triangle $\triangle ABC$ has $AB = \sqrt{10}$, $BC = \sqrt{17}$, and $CA = \sqrt{41}$. The area of $\triangle ABC$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

    Answer: 13

    Solution: Observe that $10 = 1^2 + 3^2$, $17 = 1^2 + 4^2$, $41 = 4^2 + 5^2$, and so triangle $\triangle ABC$ is congruent to triangle $\triangle A'B'C'$ given by $A' = (0, 0), B' = (1, 3), C' = (5, 4)$. The area of congruent triangles are the same, so the area of $\triangle ABC$ is the area of $\triangle A'B'D'$ plus the area of $\triangle B'C'D'$ minus the area of $\triangle A'C'D'$, or $\frac{1}{2} \cdot 5 \cdot 3 + \frac{1}{2} \cdot 4 \cdot 4 - \frac{1}{2} \cdot 5 \cdot 4 = \frac{11}{2}$. Thus, the desired answer is $\boxed{13}$.



16. Let

$$f(x) = \frac{1 + x^3 + x^{10}}{1 + x^{10}}.$$

    Compute

$$f(-20) + f(-19) + f(-18) + \cdots + f(20).$$

    Answer: 41

    Solution: We can express $f(x)$ as

$$f(x) = \frac{1 + x^3 + x^{10}}{1 + x^{10}} = 1 + \frac{x^3}{1 + x^{10}},$$

and note that
$$f(-x) = 1 + \frac{(-x)^3}{1+(-x)^{10}} = 1 - \frac{x^3}{1+x^{10}}.$$
Thus, $f(x) + f(-x) = 2$. Therefore,
$$\begin{aligned}&f(-20) + f(-19) + f(-18) + \cdots + f(20)\\ &= (f(-20) + f(20)) + (f(-19) + f(19)) + \cdots + (f(-1) + f(1)) + f(0)\\ &= 2 \cdot 20 + 1\\ &= \boxed{41}.\end{aligned}$$

17. Leanne and Jing Jing are walking around the xy-plane. In one step, Leanne can move from any point (x, y) to $(x + 1, y)$ or $(x, y + 1)$ and Jing Jing can move from (x, y) to $(x - 2, y + 5)$ or $(x + 3, y - 1)$. The number of ways that Leanne can move from $(0, 0)$ to $(20, 20)$ is equal to the number of ways that Jing Jing can move from $(0, 0)$ to (a, b), where a and b are positive integers. Compute the minimum possible value of $a + b$.

Answer: 100

Solution: Note that the only way for Leanne to get to the point $(20, 20)$ is to go from (x, y) to $(x + 1, y)$ 20 times and to $(x, y + 1)$ 20 times, and the number of ways to do this is the number of orders she can do these two kinds of 20 moves in, or $\binom{40}{20}$. Similarly, if to get to some point (a, b) Jing Jing has to go from (x, y) to $(x - 2, y + 5)$ r times and to $(x + 3, y - 1)$ s times, then the number of ways to get to this point is the number of ways to order these two kinds of $r + s$ moves, or $\binom{r+s}{r}$. Thus, we want
$$\binom{r+s}{r} = \binom{40}{20}.$$
We have that $(a, b) = (0 + r \cdot -2 + s \cdot 3, 0 + r \cdot 5 + s \cdot -1) = (-2r + 3s, 5r - s)$, so we wish to minimize $a + b = 3r + 2s$. A solution for this is $(r, s) = (20, 20)$, and we show below that this is minimal. Therefore, $(a, b) = (0 + 20 \cdot -2 + 20 \cdot 3, 0 + 20 \cdot 5 + 20 \cdot -1) = (20, 80)$. This means that $a + b = \boxed{100}$.

We now show that $(r, s) = (20, 20)$ is minimal. In other words, we show that there is no other solution (r, s) such that $3r + 2s \leq 100$. First, note that if $r + s > 40$, then $r < 20$, so then $\binom{r+s}{r}$ would contain a factor of 41, which is not okay. Thus, we need $r + s < 40$ to find another valid solution. But then we need $r > 20$. If $r \geq 23$, then we don't have enough factors of 23 in our coefficient. Thus, we only need to check cases where $r + s < 40$, $r \leq 23$, and $3r + 2s \leq 100$. Enumerating through all the solutions gives solutions that aren't equal to $\binom{40}{20}$, so $(r, s) = (20, 20)$ is the smallest solution.

18. Compute the number of positive integers $1 < k < 2021$ such that the equation
$$x + \sqrt{kx} = kx + \sqrt{x}$$
has a positive rational solution for x.

Answer: 43

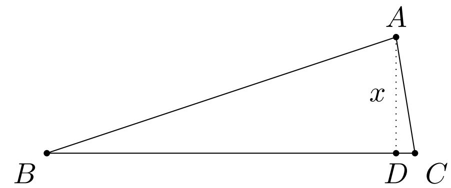
Solution: Dividing by $\sqrt{x}$, we get the equation $\sqrt{x} + \sqrt{k} = k\sqrt{x} + 1$, and rearranging gives
$$\sqrt{x}(k-1) = \sqrt{k} - 1 \implies \sqrt{x} = \frac{1}{\sqrt{k}+1}.$$

Thus, we need that $(\sqrt{k}+1)^2$ is rational. Since $(\sqrt{k}+1)^2 = k + 2\sqrt{k} + 1$ and k is an integer, we need that $\sqrt{k}$ is rational. Hence k must be a perfect square; there are $44 - 1 = \boxed{43}$ perfect squares (other than 1) under 2021, since $44^2 < 2021 < 45^2$.

19. In triangle $\triangle ABC$, point D lies on $\overline{BC}$ with $\overline{AD} \perp \overline{BC}$. If $BD = 3AD$, and the area of $\triangle ABC$ is 15, then the minimum value of AC^2 is of the form $p\sqrt{q} - r$, where p, q, and r are positive integers and q is not divisible by the square of any prime number. Compute $p + q + r$.

Answer: 250

Solution:



Let $AD = x$ and $BD = 3x$. Notice then that $BC = \frac{30}{x}$, so that $DC = \frac{30}{x} - 3x$. By the Pythagorean Theorem,

$$AC^2 = \left(\frac{30}{x} - 3x\right)^2 + x^2 = 10x^2 - 180 + \frac{900}{x^2}.$$

It suffices to minimize $10x^2 + \frac{900}{x^2}$. By the AM-GM inequality,

$$\frac{10x^2 + \frac{900}{x^2}}{2} \geq \sqrt{10x^2 \cdot \frac{900}{x^2}} = \sqrt{9000} = 30\sqrt{10},$$

so $10x^2 + \frac{900}{x^2} \geq 60\sqrt{10}$. Therefore, $AC^2 \geq 60\sqrt{10} - 180$, with $p + q + r = 60 + 10 + 180 = \boxed{250}$.

20. Suppose the decimal representation of $\frac{1}{n}$ is in the form $0.p_1p_2 \ldots p_j\overline{d_1d_2 \ldots d_k}$, where $p_1, \ldots, p_j, d_1, \ldots, d_k$ are decimal digits, and j and k are the smallest possible nonnegative integers (i.e. it's possible for $j = 0$ or $k = 0$). We define the *preperiod* of $\frac{1}{n}$ to be j and the *period* of $\frac{1}{n}$ to be k. For example, $\frac{1}{6} = 0.16666\ldots$ has preperiod 1 and period 1, $\frac{1}{7} = 0.\overline{142857}$ has preperiod 0 and period 6, and $\frac{1}{4} = 0.25$ has preperiod 2 and period 0. What is the smallest positive integer n such that the sum of the preperiod and period of $\frac{1}{n}$ is 8?

Answer: 28

Solution: Suppose $\frac{1}{n}$ has preperiod j and period k. We then can express

$$\frac{1}{n} = 0.p_1p_2 \ldots p_j\overline{d_1d_2 \ldots d_k} = \frac{a}{10^j} + \frac{b}{10^j(10^k - 1)},$$

where $a = \overline{p_1p_2 \ldots p_j}_{10}$ is the number in base 10 with digits $p_1, p_2, \cdots, p_j$, in that order and $b = \overline{d_1d_2 \ldots d_k}_{10}$ is the number in base 10 with digits $d_1, d_2, \cdots, d_k$, in that order. Thus, we have that $10^j(10^k - 1) = n(a(10^k - 1) + b)$, so $n \mid 10^j(10^k - 1)$. Note that $10^k - 1 \nmid a(10^k - 1) + b$, so n must contain some factors of $10^k - 1$. However, since k is the period, there cannot be any i such that $i < k$ and $n \mid 10^i - 1$, otherwise the period would be i. Moreover, we know that $j + k = 8$.

We now apply casework; we minimize n with respect to each (j, k). For $(j, k) = (8, 0)$, we need $n \mid 10^8$; the smallest n that satsifies this and has the preperiod equal to 8 is 2^8. For $(j, k) = (7, 1)$, we need $n \mid 10^7 \cdot 9$; the smallest the smallest n that satsifies this and has the correct preperiod is $2^7 \cdot 3$. Proceeding similarly, we determine that the smallest solution is $n = \boxed{28}$, corresponding to preperiod 2 and period 6.